Monoid Congruences, Binomial Ideals, and Their Decompositions

by

Christopher O'Neill

Department of Mathematics
Duke University

Date: ______________________

Approved:

____
Ezra Miller, Supervisor

____
Paul Aspinwall

____
Leslie Saper

____
Patricia Hersh

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2014

Abstract

Monoid Congruences, Binomial Ideals, and Their Decompositions

by

Christopher O'Neill

Department of Mathematics
Duke University

Date: ____________________

Approved:

____
Ezra Miller, Supervisor

____
Paul Aspinwall

____
Leslie Saper

____
Patricia Hersh

An abstract of a dissertation submitted in partial fulfillment of the requirements for
the degree of Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2014

Abstract

This dissertation refines and extends the theory of mesoprimary decomposition, as introduced by Kahle and Miller. The overview of the existing theory of mesoprimary decomposition for both monoid congruences and binomial ideals states all definitions and results that are relevant for subsequent chapters.

We classify mesoprimary components in both the monoid and binomial settings. Kahle and Miller give a class of redundant components in each setting that are redundant in every mesoprimary decomposition. After identifying a further class of redundant components at the level of congruences, we give a condition on the associated monoid primes that guarantees the existence of unique irredundant mesoprimary decompositions in both settings.

We introduce soccular congruences as combinatorial approximations of irreducible binomial quotients and use the theory of mesoprimary decomposition to give a combinatorial method of constructing irreducible decompositions of binomial ideals. We also demonstrate a binomial ideal that does not admit a binomial irreducible decomposition, answering a long-standing problem of Eisenbud and Sturmfels.

We extend mesoprimary decomposition of monoid congruences to congruences on monoid modules. Much of the theory extends to this new setting, including a characterization of mesoprimary monoid module congruences in terms of associated prime monoid congruences and a method for constructing coprincipal decompositions of monoid module congruences using key witnesses.

I would like to dedicate this thesis to my Mom, my Dad, and my sister Kimberly, for being there every step of the way.

Contents

List of Figures

List of Abbreviations and Symbols

Symbols

The following notation will be used throughout this document.

Q	A finitely generated commutative monoid.
$\Bbbk$	An arbitrary field.
$\Bbbk[Q]$	The monoid algebra over Q with coefficients in $\Bbbk$.
$\sim_I$	The congruence on Q induced by a binomial ideal $I \subset \Bbbk[Q]$.

Acknowledgements

First and foremost, I want to thank my advisor, Ezra Miller. Ezra has repeatedly gone above and beyond what was expected of him as my advisor. I have had the privilege of working with him every semester at Duke, from the abstract algebra courses and my qualifying exams to my eventual research projects and the writing of this dissertation. He has taught me a wealth of mathematics and has worked tirelessly to prepare me for the rest of my career, and despite an impossibly busy schedule, he has always found time to meet with me. Above all else, I am happy to call him my friend.

Next, I would like to thank my family for remaining close, even though I have been far away. The last couple of years have been hard on all of us, and I don't know what I would have done without their love and support.

I would like to thank the organizers and students in the Pacific Undergraduate Research Experience in Mathematics (PURE Math) for showing me how rewarding teaching can be. Rebecca Garcia and Bob Pelayo have worked endlessly to make this program what it is, and I am incredibly lucky to have been a part of it.

I would not have gotten as far as I have without the help of my friends at Duke. I am especially grateful to Ben Gaines and Liz Munch, who have consistently been there for me in more ways than they know. I would like to thank Sean, Diana and Elen for countless bridge nights, as well as my officemates for their uplifting conversations as we worked our way through the program. I would also like to thank

Thomas Kahle, Christine Berkesch and Sonja Mapes for their continued guidance and numerous helpful conversations over the years.

Lastly, I would like to thank the National Science Foundation, the National Security Agency, Duke University and the Mathematical Sciences Research Institute for providing me with funding during my time as a graduate student.

1

Introduction

Any ideal in a polynomial ring $S = \Bbbk[x_1, \dots, x_n]$ can be decomposed as a finite intersection of primary ideals (Noether (1921)), generalizing prime factorization of integers. These *primary decompositions* are essential tools in understanding the structure of ideals but can be difficult to compute in general. However, *monomial ideals* (ideals generated by products of variables) and *binomial ideals* (ideals whose generators are sums of at most two terms; see Definition 2.1.3) contain additional combinatorial structure that allows for concrete constructions of these decompositions (see Miller and Sturmfels (2005)). The connection between the algebraic properties of monomial ideals and their underlying combinatorial structure has been studied extensively and many concise combinatorial formulas have been found Bayer and Sturmfels (1998); Bayer et al. (1998); Hochster (1977).

Irreducible ideals are primary ideals whose quotients have simple socle; that is, upon localizing an irreducible quotient, the set of elements annihilated by the associated prime has vector space dimension 1. Every monomial ideal admits a unique irredundant *monomial irreducible decomposition*, which is an irreducible decomposition whose components are themselves monomial ideals. (Eisenbud and Sturmfels, 1996, Problem 7.3) ask if the same holds for binomial ideals, that is, whether every binomial ideal admits an irreducible decomposition whose components are themselves binomial ideals. The central result of this dissertation is Example 4.4.1, which exhibits a binomial ideal that does not admit a binomial irredudible decomposition and answers (Eisenbud and Sturmfels, 1996, Problem 7.3) in the negative. We also provide a method of constructing irreducible decompositions for any binomial ideal (Theorem 4.5.6) using its underlying combinatorial structure, though some components may not be binomial ideals.

Monomial ideals quintessentially exhibit the connection between irreducible decompositions of an ideal and its underlying combinatorial structure. Any monomial ideal I is uniquely determined by the monomials it contains, and taking the quotient S/I amounts to setting these monomials to 0. The monomials that lie outside of I naturally grade the quotient S/I with a decomposition into 1-dimensional vector spaces, and many algebraic properties can be discerned from this grading. Each $\mathfrak{p}$-primary component of the irredundant monomial irreducible decomposition for I is constructed by first locating a monomial $\mathbf{x}^w$, called a *witness*, whose annihilator modulo I equals $\mathfrak{p}$, and then constructing the monomial ideal J that contains of all monomials not lying below $\mathbf{x}^w$. These monomial witnesses are readily identified from the grading of S/I, and the resulting component J has simple socle.

Like monomial ideals, binomial ideals also possess inherent combinatorial structure. The quotient S/I of a binomial ideal I identifies, up to scalar multiple, any monomials appearing in the same binomial in I. This induces a *congruence* $\sim_I$, which is an equivalence relation that perserves additivity, on the set of monomials in S. The quotient module is naturally graded with a decomposition into 1-dimensional vector spaces, just as for monomial quotients.

Using the congruence $\sim_I$, Kahle and Miller (2013) extend the construction for monomial ideals above to the setting of binomial ideals. Given a binomial ideal

I, they pinpoint a collection of monomials in S/I that behave like witnesses (Definition 2.2.6). For each monomial witness $\mathbf{x}^w$, they construct a binomial ideal J containing I, called the *coprincipal component* at $\mathbf{x}^w$ (Definition 2.3.5), for which $\mathbf{x}^w$ is the unique greatest nonzero monomial in S/J. The resulting collection of coprincipal binomial ideals, one for each witness monomial, decomposes I (Theorem 2.3.9). Moreover, each coprincipal ideal admits a canonical primary decomposition that, if $\Bbbk$ is algebraically closed, consists entirely of binomial ideals. In this way, these *mesoprimary decompositions* (Definition 2.3.8) act as a bridge between primary components of a binomial ideal and the combinatorics of its induced congruence.

Though the quotient of the coprincipal ideal has a unique monomial in its socle, this socle may contain non-monomial elements and thus need not be simple (see Example 4.1.3). Given a coprincipal ideal I, we construct an ideal $\mathrm{Irr}(I)$, called the *irreducible closure* of I (Definition 4.5.1), that contains I and admits a canonical primary decomposition, each component of which has simple socle (Theorem 4.5.5). We also introduce *soccular congruences*, which are mesoprimary congruences with a unique key witness (Definition 4.1.4), as a combinatorial approximation of irreducible binomial quotients.

Kahle and Miller (2013) construct mesoprimary decompositions in two settings: first for monoid congruences, and then for binomial ideals. They identify a class of witnesses in each setting that always produce redundant mesoprimary components in these coprincipal decompositions. In particular, they show that one coprincipal congruence per *key witness* (Definition 2.2.6) is enough to decompose any given monoid congruence. Similarly, they show that one coprincipal ideal per *character witness* (Definition 2.3.6) is enough to decompose any given binomial ideal. However, some redundant components still remain.

In Chapter 3, we introduce the class of *true witnesses* (Definition 3.1.2), which refines the class of key witnesses, and show that one coprincipal component per true witness is enough to decompose any congruence. We classify when this resulting decomposition is irredundant (Corollary 3.2.10) and when the corresponding decomposition with one component for each character witness is irredundant (Corollary 3.3.9). Lastly, we prove the existence of a set of associated prime congruences that must appear as the associated prime congruence of some component in every mesprimary decomposition (Theorem 3.2.7).

Mesoprimary decomposition of monoid congruences is designed to parallel primary decomposition of ideals over a commutative Noetherian ring R, with *associated prime congruences* (Definition 2.2.9) playing the role of prime ideals. Just as primary decomposition of ideals in R generalizes to primary decomposition of finitely generated R-modules, mesoprimary decomposition of monoid congruences, along with its notion of associated prime congruences, generalizes to congruences on *monoid modules* (Definition 5.1.1). We extend nearly every result regarding mesoprimary decomposition of monoid congruences to congruences on finitely generated monoid modules. The resulting theory developed here gives a complete answer to (Kahle and Miller, 2013, Problem 17.11). One of the largest tasks in generalizing these results to monoid modules is to separate which constructions should happen in the monoid and which should happen in the module, since these coincide for monoid congruences.

We conclude with a collection of open problems for future study in Chapter 6.

2

Background

In this chapter, we present an overview of mesoprimary decomposition, including all definitions and results pertinent to this dissertation. Monoid congruences and binomial ideals are defined in Section 2.1, mesoprimary decomposition of monoid congruences is described in Section 2.2, and mesoprimary decomposition of binomial ideals is covered in Section 2.3.

2.1 Monoid congruences, binomial ideals, and primary decomposition

We begin by defining monoid congruences and binomial ideals, both of which play a central role in the results that follow.

Definition 2.1.1. A *commutative monoid* $(Q, +)$ is a set Q with a binary operation $+$ that is commutative, associative, and has an identity element $0 \in Q$. A subset $T \subseteq Q$ is an *ideal* if $Q + T \subset T$, and an ideal T is *prime* if $Q \backslash T$ is closed under $+$. We write

$$\langle q_1, \ldots, q_k \rangle = \bigcup_{i=1}^{k} (q_i + Q)$$

to denote the ideal generated by the elements $q_1, \dots, q_k \in Q$.

Definition 2.1.2. A *congruence* on a monoid Q is an equivalence relation $\sim$ on Q such that $a + c \sim b + c$ whenever $a \sim b$ for $a, b, c \in Q$. The *common refinement* $\sim \cap \approx$ of two congruences $\sim$ and $\approx$ on Q is the congruence that relates a and b in Q if and only if both $a \sim b$ and $a \approx b$.

The condition for an equivalence relation $\sim$ on Q to be a congruence ensures that the set $Q/\sim$ of equivalence classes under $\sim$ has a well-defined monoid structure inherited from Q.

Definition 2.1.3. Fix a monoid Q and a field $\Bbbk$. The *monoid algebra over Q with coefficients in* $\Bbbk$, denoted $\Bbbk[Q]$, is the set of finite formal sums of elements of the form $c\mathbf{x}^q$ for $q \in Q$ and $c \in \Bbbk$, that is,

$$\Bbbk[Q] = \bigoplus_{q \in Q} \Bbbk \cdot \mathbf{x}^q = \{c_1\mathbf{x}^{q_1} + \cdots + c_k\mathbf{x}^{q_k} : q_1, \dots, q_k \in Q, c_1, \dots, c_k \in \Bbbk\},$$

where multiplication is given by $\mathbf{x}^a \cdot \mathbf{x}^b = \mathbf{x}^{a+b}$ for $a, b \in Q$ and the distributive property. A *monomial* in $\Bbbk[Q]$ is an element of the form $\mathbf{x}^q \in \Bbbk[Q]$ for $q \in Q$, and a *binomial* is an element of the form $\mathbf{x}^a + \lambda\mathbf{x}^b \in \Bbbk[Q]$ for $a, b \in Q$, $\lambda \in \Bbbk$. In particular, any monomial is a binomial by taking $\lambda = 0$. A *monomial ideal* is an ideal in $\Bbbk[Q]$ generated by monomials, and a *binomial ideal* is an ideal generated by binomials.

Remark 2.1.4. A binomial ideal $I \subset \Bbbk[Q]$ induces a congruence $\sim_I$ on Q that sets $a \sim_I b$ whenever $\mathbf{x}^a + c\mathbf{x}^b \in I$ for some nonzero $c \in \Bbbk$.

The following example appeared as (Kahle and Miller, 2013, Example 2.19).

Example 2.1.5. Let $I = \langle x^2 - xy, xy - y^2, x^3 \rangle \subset \Bbbk[x, y]$. Figure 2.1 depicts the congruence $\sim_I$ on $Q = \mathbb{N}^2$ induced by I. For instance, since $x^2 - xy \in I$, the elements $(2, 0)$ and $(1, 1)$ in Q that correspond to the monomials x^2 and xy in $\Bbbk[x, y]$

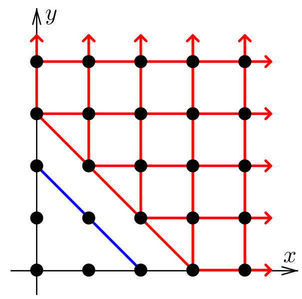


FIGURE 2.1: The congruence induced by $\langle x^2 - xy, xy - y^2, x^3 \rangle \subset \Bbbk[x,y]$ on $\mathbb{N}^2$.

are identified under $\sim_I$. Additionally, since the monomials x^3 and x^3y both lie in I, so does $x^3 - x^3y$, so the elements $(3,0)$ and $(3,1)$ in Q corresponding to these monomials are identified under $\sim_I$. The quotient $Q/{\sim}$ has 5 elements: one for each of the elements $(0,0)$, $(1,0)$ and $(0,1)$ in Q that are singleton under $\sim$; the equivalence class containing $(2,0)$, $(1,1)$ and $(0,2)$; and the class containing the exponent vector of every monomial in I.

Definition 2.1.6. An element $q \in Q$ is a *unit* if it has an inverse in Q. An element $q \in Q$ is *nil* if $q + a = q$ for all $a \in Q$.

Remark 2.1.7. A nil element in a monoid Q is unique when it exists, and we often use ∞ to denote the nil element. If $I \subset \Bbbk[Q]$ is a binomial ideal that contains monomials, the set of monomials in I forms a single equivalence class in $\sim_I$, and this class is nil in the quotient $Q/{\sim_I}$.

Some properties of monoid elements hold up to unit multiple, so we now give notation for this. Lemma 2.1.9 appeared as (Grillet, 2001, Proposition I.4.1).

Definition 2.1.8. Fix a monoid Q. *Green's preorder* on Q is the divisibility preorder that sets $p \preceq q$ whenever $\langle p \rangle \supset \langle q \rangle$. *Green's relation* on Q is the equivalence relation that sets $p \sim q$ whenever $\langle p \rangle = \langle q \rangle$.

Lemma 2.1.9. *Green's equivalence $\sim$ on a monoid Q is a congruence, and the quotient $Q/\sim$ is partially ordered by divisibility.*

We now review primary decomposition for Noetherian rings. For a more thorough treatment, see (Eisenbud, 1995, Chapter 3).

Definition 2.1.10. Fix a Noetherian ring R and a finitely generated R-module M.

1. A prime ideal $\mathfrak{p} \subset R$ is *associated to* M if $\operatorname{ann}(m) = \mathfrak{p}$ for some $m \in M$.

2. The set of associated primes of M is denoted $\operatorname{Ass}(M)$.

3. A submodule $N \subset M$ is $\mathfrak{p}$-*primary* if $\operatorname{Ass}(M/N) = \{\mathfrak{p}\}$.

4. If $I \subset R$ is an ideal, then $\operatorname{Ass}(I)$ denotes the associated primes of R/I.

5. A submodule $N \subset M$ is *irreducible* if whenever $N = N_1 \cap N_2$ for submodules $N_1, N_2 \subset M$, we have $N = N_1$ or $N = N_2$.

6. An expression of the form $N = N_1 \cap \cdots \cap N_r$ for a submodule $N \subset M$ is a *primary decomposition* (resp. an *irreducible decomposition*) if each N_i is primary (resp. irreducible).

Theorem 2.1.11 implies that every submodule of a finitely generated module over a Noetherian ring admits a primary decomposition.

Theorem 2.1.11. *Fix a Noetherian ring R and a finitely generated R-module M.*

1. $\operatorname{Ass}(M)$ *is nonempty and finite.*

2. *Any irreducible submodule of M is primary.*

3. *The intersection of any two $\mathfrak{p}$-primary submodules of M is $\mathfrak{p}$-primary.*

4. *Every submodule of M admits an irreducible decomposition.*

2.2 Mesoprimary decomposition of monoid congruences

We begin this section by introducing primary congruences and mesoprimary congruences, which we use to decompose arbitrary monoid congruences.

Definition 2.2.1. A monoid Q is

1. *primary* if every element $q \in Q$ is either *cancellative* (meaning $a + q = b + q$ implies $a = b$ for any $a, b \in Q$) or *nilpotent* (meaning some integer multiple of q is nil in Q), and

2. *mesoprimary* if it is primary and every non-nil $q \in Q/\sim$ is *partly cancellative* (meaning $a + q = b + q$ implies $a = b$ for cancellative $a, b \in Q$).

A congruence $\sim$ on Q is *primary* (resp. *mesoprimary*) if the monoid $Q/\sim$ is primary (resp. mesoprimary).

The following example, which appeared as (Kahle and Miller, 2013, Example 1.1), demonstrates the difference between primary and mesoprimary congruences.

Example 2.2.2. Let $I = \langle y(x^2 - 1), y^2(x - 1), y^3\rangle \subset \Bbbk[x, y]$. The congruence $\sim_I$ induced on $Q = \mathbb{N}^2$ (see Figure 2.2) is primary, since every element on the x-axis is cancellative and any element with nonzero y-coordinate is nilpotent. However, $\sim_I$ is not mesoprimary, since (for instance) $(0, 0) + (0, 1) \sim_I (2, 0) + (0, 1)$, which means $(0, 1)$ is not partly cancellative. On the other hand, the ideals $\langle y\rangle$, $\langle x^2 - 1, y^2\rangle$ and $\langle x - 1, y^3\rangle$ in $\Bbbk[x, y]$ all induce mesoprimary congruences (see Figure 2.3).

Remark 2.2.3. It is important to note that the notion of primary congruence differs from that of a primary binomial ideal. For instance, the ideal in Example 2.2.2 induces a primary congruence, but is not primary over any field of characteristic 0.

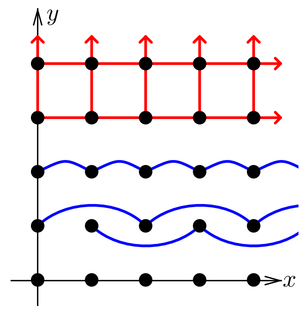


FIGURE 2.2: The congruence induced by $\langle y(x^2-1), y^2(x-1), y^3\rangle \subset \Bbbk[x,y]$ on $\mathbb{N}^2$.

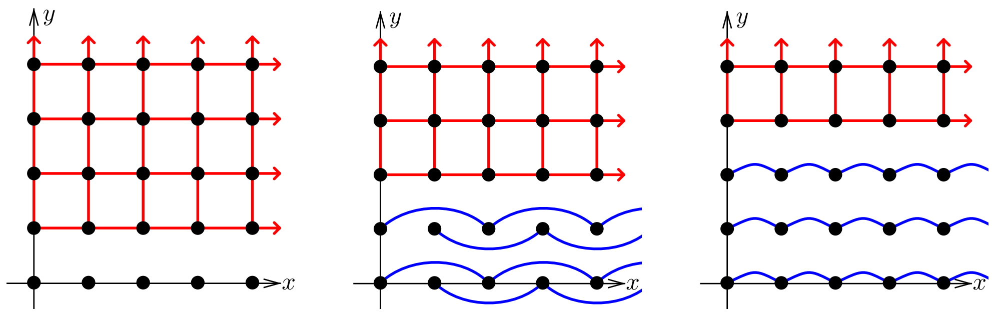


FIGURE 2.3: The congruences induced by the ideals $\langle y\rangle$ (left), $\langle x^2-1, y^2\rangle$ (middle) and $\langle x-1, y^3\rangle$ (right) in $\Bbbk[x,y]$ on $\mathbb{N}^2$.

Lemma 2.2.4 appeared as (Kahle and Miller, 2013, Lemma 2.19) and was central to several results, including Theorems 2.2.10 and 2.2.15.

Lemma 2.2.4. *Fix a primary monoid Q, and let $F \subset Q$ denote the submonoid of cancellative elements. The quotient monoid Q/F defined by the congruence*

$$p \sim q \Leftrightarrow p + f = q + g \text{ for some } f, g \in F$$

is partially ordered by divisibility, and every non-identity element is nilpotent. Moreover, if Q is finitely generated, then Q/F is finite.

Definition 2.2.5. The *localization of Q at P*, denoted Q_P, is the set $Q \times F$ for $F = Q \backslash P$ modulo the equivalence relation that relates (q, f) and (q', f') whenever $w + q + f' = w + q' + f$ for some $w \in Q$. The localization Q_P is naturally a monoid, and we often denote by $q - f$ the element $(q, f) \in Q_P$. Any congruence $\sim$ on Q induces a congruence on Q_P, and we often write $\overline{Q}_P$ to denote $Q_P/{\sim}$.

We are now ready to define witnesses, which are the analogues of elements of a module over a Noetherian ring whose annihilator is a prime ideal. Witnesses detect which objects are associated to a given congruence (see Theorems 2.2.8 and 2.2.10) and are used to construct coprincipal decompositions (see Theorems 2.2.15 and 2.3.9). Witnesses were defined in (Kahle and Miller, 2013, Definition 4.7), and aides were defined in (Kahle and Miller, 2013, Definition 4.10).

Definition 2.2.6. Fix a monoid Q, a prime $P \subset Q$, and a congruence $\sim$. Let $\overline{Q} = Q_P/{\sim}$, and for $q \in Q$, let $\overline{q}$ denote the image of q in $\overline{Q}$.

1. An element $q \in Q$ is an *aide* for an element $w \in Q$ and a generator $p \in P$ if $\overline{w} \neq \overline{q}$, $\overline{w} + \overline{p} = \overline{q} + \overline{p}$, and $\overline{q}$ is maximal under divisibility in the set $\{\overline{q}, \overline{w}\}$. We say q is a *key aide* for w if q is an aide for w for each generator of P.

2. An element $w \in Q$ is a P*-witness for* $\sim$ if it has an aide for each $p \in P$, and a P-witness w for $\sim$ is *key* if it has a key aide.

3. The prime P is *associated to* $\sim$ if there exists a P-witness for $\sim$.

Example 2.2.7. Consider the congruence $\sim_I$ induced by the ideal I from Example 2.2.2, and let $P = \langle (0,1) \rangle \subset Q = \mathbb{N}^2$. Each $(k, 0) \in Q$ is a key P-witness for $\sim_I$, since $(k, 0) + p \sim_I (k + 2, 0) + p$ for any $p \in P$. Similarly, $(0, 1)$ and $(1, 1)$ are key P-witnesses for $\sim$, and each is a key aide for the other. Lastly, $(0, 2)$ is a key P-witness for $\sim$ with key aide ∞.

Theorem 2.2.8, which originally appeared as (Kahle and Miller, 2013, Corollary 4.21), gives an equivalent condition for a congruence to be primary that justifies this choice of vocabulary.

Theorem 2.2.8. *Fix a monoid Q. A congruence $\sim$ on Q is primary if and only if it has exactly one associated monoid prime.*

We now define the associated objects of mesoprimary congruences (Definition 2.2.9), which play the role of the associated object for mesoprimary congruences as prime ideals do for primary congruences. Theorem 2.2.10, which gives an alternative characterization of mesoprimary congruences in terms of the action of their cancellative submonoids, originally appeared as (Kahle and Miller, 2013, Theorems 6.1 and 6.7).

Definition 2.2.9. Fix a congruence $\sim$ on a monoid Q, a prime $P \subset Q$, and an element $q \in Q$ that is non-nil under $\sim$.

1. Let $G_P \subset Q_P$ denote the unit group of Q_P, and let $K_q^P \subset G_P$ denote the stabilizer of $\overline{q} \in \overline{Q}_P$ under the action of G_P.
2. Let $\approx$ denote the congruence on Q_P that sets $a \approx b$ whenever
   (a) a and b lie in P_P, or
   (b) a and b lie in G_P and $a - b \in K_q^P$.
3. The *P-prime congruence* of $\sim$ at q is given by $\ker(Q \to Q_P/\approx)$.
4. The P-prime congruence at q is *associated to* $\sim$ if q is a key witness for P.

Theorem 2.2.10. *Fix a monoid Q and a congruence $\sim$ on Q, and let $\overline{Q} = Q/\sim$. The following are equivalent:*

1. $\sim$ is mesoprimary.

2. $\sim$ has exactly one associated prime congruence.

3. $\sim$ is primary with associated prime $P \subset Q$, and for $F = \overline{Q}\setminus\overline{P}$ and $T = \overline{Q}\setminus\{\infty\}$, the maps $T \xrightarrow{+f} T$ and $F \xrightarrow{+t} T$ are injective for each $f \in F$ and $t \in T$.

Definition 2.2.11 appeared as (Kahle and Miller, 2013, Definitions 7.1 and 7.2) and introduces cogenerators as maximal non-nil elements. Given a congruence $\sim$ on Q and a witness $w \in Q$ for $\sim$, Definition 2.2.12, which appeared as (Kahle and Miller, 2013, Definition 7.7), gives a mesoprimary congruence that coarsens $\sim$ and has w as its only cogenerator. These congruences are used in Theorem 2.2.15 to explicitly construct a mesoprimary decomposition for $\sim$.

Definition 2.2.11. A key witness w for a congruence $\sim$ on Q is a *cogenerator* for $\sim$ if $q + p$ is nil modulo $\sim$ for all $p \in P$. The congruence $\sim$ is *P-coprincipal* if it is P-mesoprimary and every cogenerator for $\sim$ generates the same ideal modulo $\sim$.

Definition 2.2.12. Fix a congruence $\sim$ on Q, a prime ideal $P \subset Q$, and a witness $w \in Q$ for $\sim$. The congruence *cogenerated by w along P*, denoted $\sim_w^P$, is the coprincipal congruence that relates $a, b \in Q$ if and only if one of the following is satisfied:

1. the ideal generated by $\overline{a}$ and $\overline{b}$ does not contain $\overline{w}$ in $\overline{Q}_P$;

2. $\overline{a}$ and $\overline{b}$ differ by a unit in $\overline{Q}_P$ and $\overline{a} + \overline{c} = \overline{b} + \overline{c} = \overline{w}$ for some $\overline{c} \in \overline{Q}_P$.

Example 2.2.13. In Example 2.2.2, the mesoprimary congruences on $\mathbb{N}^2$ induced by the ideals $\langle y \rangle$, $\langle x^2 - 1, y^2 \rangle$ and $\langle x - 1, y^3 \rangle$ are all coprincipal components of the congruence induced by $I = \langle y(x^2 - 1), y^2(x - 1), y^3 \rangle$. Indeed, each is mesoprimary and has all of its cogenerators in the same Green's class, and each is constructed by taking a witness w for $\sim_I$, identifying everything not below w with the nil, and identifying elements below w so that each prime congruence matches that at w.

Definition 2.2.14 appeared as (Kahle and Miller, 2013, Definition 8.1).

Definition 2.2.14. Fix a congruence $\sim$ on Q. An expression of $\sim$ as the common refinement $\bigcap_i \approx_i$ of mesoprimary congruences is a *mesoprimary decomposition* if, for each $\approx_i$ with associated prime $P_i \subset Q$, the P_i-prime congruences of $\sim$ and $\approx_i$

at each cogenerator for $\approx_i$ coincide. A mesoprimary decomposition is *key* if every cogenerator for each $\approx_i$ is a key witness for $\sim$.

Theorem 2.2.15 appeared as (Kahle and Miller, 2013, Theorem 8.4).

Theorem 2.2.15. *Every congruence $\sim$ on a monoid Q is the common refinement of the coprincipal components cogenerated by its key witnesses.*

We conclude this section by stating as Corollary 2.2.16 a consequence of Kahle and Miller's proof of Theorem 2.2.15, which will be used in the proof of Theorem 4.2.5.

Corollary 2.2.16. *Given a congruence $\sim$ on Q and elements $a, b \in Q$ with $a \not\sim b$, there exists a $P \subset Q$ and $u \in Q$ such that (after possibly swapping a and b) $a + u$ is a key P-witness with key aide $b + u$.*

Example 2.2.17. Resuming the discussion from Example 2.2.2, the congruences in Figure 2.3 form a mesoprimary decomposition for the congruence $\sim$ in Figure 2.2. Indeed, each of the mesoprimary congruences is a coprincipal component of some witness, and each of the key witnesses in Example 2.2.7 is the cogenerator of one of these components. By Theorem 2.2.15, the common refinement of these congruences is $\sim$.

2.3 Mesoprimary decomposition of binomial ideals

We begin this section by defining the associated mesoprime ideals of a given binomial ideal, which are the binomial analogues of associated prime congruences.

Definition 2.3.1. Given a monoid prime $P \subset Q$, the *monomial localization of* $\Bbbk[Q]$ *along* P, denoted $\Bbbk[Q]_P$, is obtained by adjoining to $\Bbbk[Q]$ the inverses of all monomials outside of the monomial ideal $\mathfrak{m}_P = \langle \mathbf{x}^p : p \in P \rangle$.

Definition 2.3.2. Fix a binomial ideal $I \subset \Bbbk[Q]$, a prime $P \subset Q$, and $q \in Q$ with $\mathbf{x}^q \notin I_P$. Let $G_P \subset Q_P$ denote the unit group of Q_P, and let $K_q^P \subset G_P$ denote the subgroup of G_P that fixes the class of q modulo $\sim_I$. Let $\rho : K_q^P \to \Bbbk^*$ denote the group homomorphism such that $\mathbf{x}^u - \rho(u)$ lies in the kernel of the $\Bbbk[G_P]$-module homomorphism $\Bbbk[G_P] \to \Bbbk[Q_P]/I_P$ taking $1 \mapsto \mathbf{x}^q$. The *P-mesoprime ideal of I at q* is the preimage I_q^P in $\Bbbk[Q]$ of the ideal

$$(I_q^P)_P = \langle \mathbf{x}^u - \rho(u-v)\mathbf{x}^v : u - v \in K_q^P \rangle + \mathfrak{m}_P \subset \Bbbk[Q]_P.$$

The mesoprime I_q^P is *associated to I* if q is a P-witness for I.

We now define mesoprimary binomial ideals as in (Kahle and Miller, 2013, Definition 10.4). Theorem 2.3.4, which first appeared as (Kahle and Miller, 2013, Proposition 12.8), is the binomial analogue of Theorem 2.2.10 and states that mesoprimary ideals are precisely the binomial ideals that have a unique associated mesoprime.

Definition 2.3.3. A binomial ideal $I \subset \Bbbk[Q]$ is *mesoprimary* if $\sim_I$ is mesoprimary and I is maximal among binomial ideals inducing the congruence $\sim_I$.

Theorem 2.3.4. *A binomial ideal $I \subset \Bbbk[Q]$ is mesoprimary if and only if it has exactly one associated mesoprime.*

We now define the binomial coprincipal components used to construct mesoprimary decompositions of arbitrary binomial ideals in Theorem 2.3.9.

Definition 2.3.5. Fix a binomial ideal $I \subset \Bbbk[Q]$, a prime $P \subset Q$, and a witness $w \in Q$. The *P-coprincipal component of I at w* is the preimage $W_w^P(I) \subset \Bbbk[Q]$ of the ideal $I_P + I_{\rho,P} + M_w^P(I) \subset \Bbbk[Q]_P$, where $I_{\rho,P}$ is the P-mesoprime ideal at w and $M_w^P(I)$ is the ideal generated by monomials $\mathbf{x}^u \in \Bbbk[Q]$ such that $w \notin \langle u \rangle \subset Q_P$.

When constructing the coprincipal decomposition of a monoid congruence $\sim$ in Theorem 2.2.15, it suffices to take the common refinement of the coprincipal components cogenerated by the key witnesses for $\sim$. Likewise, when constructing a coprincipal decomposition for a binomial ideal I, it suffices to intersect the components cogenerated by character witnesses for I. Definition 2.3.6 appeared as (Kahle and Miller, 2013, Definition 16.3).

Definition 2.3.6. Fix a binomial ideal $I \subset \Bbbk[Q]$, a prime $P \subset Q$.

1. The *P-cellular component of I* is the preimage $C_P(I) \subset \Bbbk[Q]$ of the sum $I_P + M^P_{\mathbf{w}}(I)$, where $\mathbf{w}$ is the set of all I-witnesses for P and $M^P_{\mathbf{w}}(I)$ is the ideal generated by the monomials $\mathbf{x}^u \in \Bbbk[Q]$ such that $\mathbf{w} \cap \langle u \rangle = \varnothing$.

2. The *testimony of an I-witness w at P* is the set $T_P(w)$ of mesoprimes $C_P(I)^P_{w+p}$ in the P-cellular component of I, one for each generator $p \in P$.

3. An I-witness w for P is a *character witness* if w is maximal among $C_P(I)$-witnesses for P or if the intersection of the mesoprimes in its testinomy $T_P(w)$ properly contains I^P_w.

Example 2.3.7. It is important to note that character witnesses differ from key witnesses. Let $I = \langle x(z-1), y(w-1), x^2, xy, y^2 \rangle \subset \Bbbk[x, y, z, w]$. The induced congruence $\sim_I$ is primary to the monoid prime P such that $\mathfrak{m}_P = \langle x, y \rangle$. The origin $\mathbf{0} \in \mathbb{N}^4$ is a witness for I that is not key. However, its testimony is

$$T_P(\mathbf{0}) = \{\langle z-1, x, y \rangle, \langle w-1, x, y \rangle\}.$$

Since the intersection of these mesoprime ideals is not a binomial ideal, $\mathbf{0}$ is a character witness for I.

Definition 2.3.8 appeared as (Kahle and Miller, 2013, Definitions 13.1 and 16.3), and is the binomial analogue of Definition 2.2.14.

Definition 2.3.8. An expression for a binomial ideal $I \subset \Bbbk[Q]$ as an intersection $\bigcap_j I_j$ of mesoprimary ideals is a *mesoprimary decomposition* if, for each component I_j

with associated prime $P_j \subset Q$, the P_j-mesoprime of I and I_j at each cogenerator for I_j coincides. A mesoprimary decomposition for I is *combinatorial* (resp. *characteristic*) if every cogenerator for each component is a witness (resp. character witness) for I.

We are now ready to state Theorem 2.3.9, which originally appeared as (Kahle and Miller, 2013, Theorems 13.4 and 16.9). In particular, it implies that every binomial ideal in a monoid algebra admits a mesoprimary decomposition.

Theorem 2.3.9. *Any binomial ideal $I \subset \Bbbk[Q]$ is the intersection of the coprincipal components cogenerated by its character witnesses.*

We conclude the section by stating (Kahle and Miller, 2013, Theorems 15.6 and 15.9). This, together with Theorem 2.3.9, gives a method to construct a primary decomposition for any binomial ideal $I \subset \Bbbk[Q]$. Moreover, if $\Bbbk$ is algebraically closed, each component in the resulting primary decomposition for I is also a binomial ideal.

Theorem 2.3.10. *Any mesoprimary ideal $I \subset \Bbbk[Q]$ admits a canonical minimal primary decomposition $I = \bigcap_i I_i$, and when $\Bbbk$ is algebraically closed, each component of this decomposition is binomial.*

3

Irredundant mesoprimary decompositions

At the heart of mesoprimary decomposition, both for monoid congruences and for binomial ideals, lies a notion of associated objects analogous to associated prime ideals in standard primary decomposition of Noetherian rings (Eisenbud, 1995, Chapter 3). In particular, any congruence $\sim$ on Q has a collection of associated prime congruences, and each component in a mesoprimary decomposition for $\sim$ has precisely one associated prime congruence. Similarly, any binomial ideal $I \subset \Bbbk[Q]$ has a collection of associated mesoprime ideals. However, unlike standard primary decomposition, eliminating redundant mesoprimary components in either setting can produce decompositions in which some of the associated objects do not appear as the associated object of any component (Example 3.1.1).

In this chapter, we identify a class of witnesses in each setting whose associated objects each appear as the associated object of some component in every mesoprimary decomposition, completing the theory of mesoprimary decomposition as a more faithful analog of primary decomposition. Using these witnesses, we characterize which congruences admit a unique irredundant decomposition (that is, no component can be omitted) and a unique minimal decomposition (that is, one component per truly associated object). Lastly, we characterize which components in any given coprincipal decomposition for given a binomial ideal can be omitted to obtain a coprincipal decomposition for its induced congruence.

3.1 True witnesses of monoid congruences

Key witnesses form a restricted class of witnesses sufficient for decomposing monoid congruences. However, coprincipal components cogenerated by key witnesses may still be redundant, as Example 3.1.1 demonstrates. This motivates the definition of true witness in Definition 3.1.2.

Example 3.1.1. Let $I = \langle x^3 - xy^2, x^3(z-1), x^2y - y^3, y^3(w-1), x^4, y^4\rangle \subset \Bbbk[x, y, z, w]$. Its congruence $\sim_I$ on $Q = \mathbb{N}^4$ is depicted in Figure 3.1, projected onto the xy-plane. This congruence is primary to $P \subset Q$ for $\mathfrak{m}_P = \langle x, y\rangle$ and has five Green's classes of key witnesses, namely those containing the monomials x^2, y^2, x^3, y^3, and x^3y. Indeed, each of x^2 and y^2 is a key aide for the other, wx^3 is a key aide for x^3, zy^3 is a key aide for y^3, and x^3y has nil as its key aide. Of these, x^2 and y^2 yield redundant components in the coprincipal decomposition for $\sim_I$ in Theorem 2.2.15, and the remaining three form an irredundant mesoprimary decomposition for $\sim_I$. On the other hand, all five are character witnesses for I, and none may be omitted from the coprincipal decomposition for I in Theorem 2.3.9.

Definition 3.1.2. Fix a congruence $\sim$ on Q, and a prime $P \subset Q$.

1. A *P-cover congruence* for a witness $w \in Q$ is the P-prime congruence at a non-nil element $w + p$ for some generator p of P.

2. The *discrete testimony* of an element $w \in Q$ at P is the set $T_P(w)$ of P-cover congruences of w. The discrete testimony of w is *suspicious* if the common

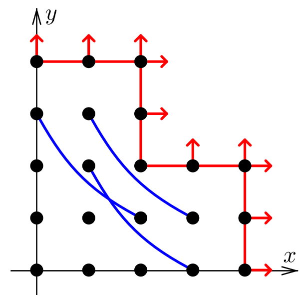


FIGURE 3.1: A congruence $\sim$ on $\mathbb{N}^4$ with key witnesses whose coprincipal components are redundant, projected onto the xy-plane.

refinement of the P-cover congruences in the testimony coincides with the P-prime congruence at w.

3. The element $w \in Q$ is a *true witness* if it is maximal among P-witnesses for $\sim$ or if its discrete testimony is not suspicious.

4. A prime congruence $\approx$ is *truly associated* to $\sim$ if it is the prime congruence at a true witness for $\sim$.

True witnesses are the analogues of character witnesses for congruences, except that their testimony is computed by refining congruences instead of intersecting ideals. We now give an equivalent condition for identifying true witnesses. As an immediate consequence, Corollary 3.1.5 shows that true witnesses are key witnesses, a fact that fails for their binomial analogues; see Remark 3.3.2.

Proposition 3.1.3. *Fix a congruence $\sim$ on Q. The discrete testimony of a P-witness w for $\sim$ is not suspicious if and only if w has a key aide w' that is either nil or generates the same ideal as w in Q_P.*

Proof. If w has ∞ as a key aide, then its discrete testimony is empty. If w has a key aide w' in its Green's class in Q_P, then each prime congruence in its discrete testimony identifies w and w', and thus so does their common refinement. Either way, the discrete testimony of w is not suspicious.

Now suppose the discrete testimony of w is not suspicious and that ∞ is not a key aide. The set $T_P(w)$ is thus nonempty, and the common refinement of the prime congruences in $T_P(w)$ relates some u and v outside of P that are not related under the prime congruence $\approx$ at w. This means any element w' with $w + u = w' + v$ must satisfy $w + p = w' + p$ for each $p \in P$, which means w' is a key aide for w. $\square$

Corollary 3.1.4. *The element w in Proposition 3.1.3 is a true witness if and only if w is maximal among P-witnesses for $\sim$ or it has a key aide that generates the same ideal as w in Q_P.* $\square$

Corollary 3.1.5. *Every true witness for a given congruence is a key witness.* $\square$

We now give the main result for this section. Theorem 3.1.6 shows that when constructing an induced coprincipal decomposition for a given congruence, it suffices to consider true witnesses. In particular, in the decomposition given in Theorem 2.2.15, any component cogenerated by a non-true witness is redundant and can be omitted.

Theorem 3.1.6. *Every congruence on Q is the common refinement of the coprincipal congruences cogenerated by its true witnesses.*

Proof. Fix a congruence $\sim$ on Q and a key P-witness w that is not true. We wish to show that the congruence $\sim_w^P$ is redundant in the coprincipal decomposition $\sim = \bigcap_i \sim_i$ in Theorem 2.2.15. Fix $q, q' \in Q$ not identified under $\sim_w^P$. We produce a component $\sim_j$ that does not relate q and q'. It suffices to assume that $Q = Q_P$ so that P is maximal. First, suppose q and q' lie in distinct Green's classes in Q. Since w is not true, it is not maximal, so some maximal P-witness v lies above w. The nil class of the coprincipal congruence $\sim_v^P$ is properly contained in the nil class of $\sim_w^P$, so q and q' are not both nil under $\sim_v^P$. Furthermore, outside of its nil class, $\sim_v^P$ does not relate any elements that lie in separate Green's classes. In particular, $\sim_v^P$ does not relate q and q'.

Now suppose q and q' lie in the same Green's class in Q_P. Since q and q' are not both nil modulo $\sim_w^P$, there exists $u \in Q$ such that $q + u$ and $q' + u$ are in the same Green's class as w, Furthermore, any component that does not relate $q+u$ and $q'+u$ will not relate q and q', so replacing q with $q + u$ and q' with $q' + u$, it suffices to take $u = 0$ and $q' = w$. Since w is not a true witness, q is not a key aide for w, so $w + p \not\sim q + p$ for some generator $p \in P$. This means some component $\sim_j$ does not relate $w + p$ and $q + p$, and thus does not relate w and q, as desired. □

3.2 Irredundant mesoprimary decompositions of congruences

In this section, we investigate minimal and irredundant coprincipal decompositions of monoid congruences (Definition 3.2.1). We also prove that each truly associated prime congruence (Definition 3.1.2) of a given congruence $\sim$ on a monoid Q appears as the associated prime congruence of some mesoprimary component in every mesoprimary decomposition for $\sim$.

We first define the types of mesoprimary decompositions of interest in this section.

Definition 3.2.1. A mesoprimary decomposition $\bigcap_i \sim_i$ of a congruence $\sim$ is

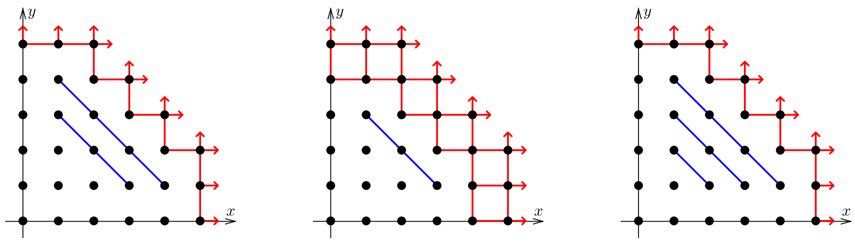
1. *minimal* if $\sim_i$ and $\sim_j$ have distinct associated prime congruences for $i \neq j$;

2. *irredundant* if no $\sim_i$ can be omitted;

3. *coprincipal* if each $\sim_i$ is coprincipal;

4. *induced* if each $\sim_i$ is a common refinement of coprincipal components.

Remark 3.2.2. The coprincipal component $\sim_w^P$ of a congruence $\sim$ on Q at a P-witness w in Definition 2.2.12 is determined by the congruence $\sim$. More precisely, it is the finest coprincipal congruence with cogenerator w that can appear in a mesoprimary decomposition for $\sim$. It is for this reason that, for the purpose of minimality,



$$I \quad = \quad I_1 \quad \cap \quad I_2$$

FIGURE 3.2: A mesoprimary decomposition of $I = \langle x^3y - x^2y^2, x^2y^2 - xy^3, x^5, y^5\rangle$ into $I_1 = \langle x^3y - x^2y^2, x^2y^2 - xy^3, x^4, y^4\rangle$ and $I_2 = \langle x^2y - xy^2, x^5, y^5\rangle$ in $\Bbbk[x, y]$. The first component is cogenerated by a witness for $\sim_I$ that is not key, but neither component is redundant.

we restrict our attention to induced mesoprimary decompositions. Indeed, if the induced condition is relaxed, even coprincipal components whose cogenerator is a non-key witness may be essential, as Example 3.2.3 demonstrates (see Problem 6.1).

Example 3.2.3. The ideal $I = \langle x^3y - x^2y^2, x^2y^2 - xy^3, x^5, y^5\rangle$ is the intersection of $I_1 = \langle x^3y - x^2y^2, x^2y^2 - xy^3, x^4, y^4\rangle$ and $I_2 = \langle x^2y - xy^2, x^5, y^5\rangle$. Their congruences $\sim$, $\sim_1$, and $\sim_2$, respectively, are depicted in Figure 3.2. Both $\sim$ and $\sim_2$ are coprincipal with cogenerator $(4, 1)$, but $\sim_2$ is not the coprincipal component cogenerated by $(4, 1)$ since it also identifies $(2, 1)$ and $(1, 2)$. As such, this mesoprimary decomposition is not induced. Additionally, $\sim_1$ is cogenerated by a non-key non-character witness for $\sim$, but neither component of this mesoprimary decomposition can be omitted.

An important observation is that any witness whose discrete testimony is not suspicious must appear as a cogenerator in any mesoprimary decomposition. We record this in Lemma 3.2.4, which serves as the foundation for the major results in this section.

Lemma 3.2.4. *Fix a mesoprimary decomposition $\bigcap_i \sim_i$ for a congruence $\sim$ on Q, and a P-witness w. If the discrete testimony of w is not suspicious, then w is a cogenerator for some $\sim_i$.*

Proof. Let $\approx$ denote the P-prime congruence at w, and let $\approx_i$ denote the prime congruence associated to $\sim_i$ for each i. By Proposition 3.1.3, either w has ∞ as a key aide, or w has a key aide w' that is Green's equivalent to w in the localization Q_P. If w has ∞ as a key aide, then it is a cogenerator for $\sim$, so any mesoprimary component $\sim_i$ under which w is not nil also has w as a cogenerator.

Alternatively, suppose w has a key aide w' in the same Green's class as w in Q_P. Since $w \neq w'$, some mesoprimary component $\sim_i$ does not relate w and w'. Neither w nor w' is nil under $\sim_i$, but for each generator p of P, the prime congruence at $w + p$ relates w and w'. This means each $w + p$ must be nil under $\sim_i$ because $\sim_i$ is mesoprimary, so w is a cogenerator for $\sim_i$. $\square$

The symmetry in Example 3.2.5 demonstrates that in some situations it is impossible to eliminate all redundancy without making arbitrary choices.

Example 3.2.5. Let $I = \langle x^2 - xy, xy - y^2\rangle \subset \Bbbk[x, y]$. The congruence $\sim_I$ has two associated monoid primes, namely the maximal ideal P and the empty ideal $\varnothing$. Theorem 2.3.9 produces the coprincipal decomposition

$$I = \langle x^2 - xy, xy - y^2\rangle = \langle x^2, y\rangle \cap \langle x, y^2\rangle \cap \langle x - y\rangle.$$

The third component is $\varnothing$-primary, and the first two components are P-primary. Either, but not both, of the first two components can be omitted without affecting the intersection. However, each is cogenerated by a true witness for $\sim_I$.

The following lemma is central to the proof of Theorem 3.2.7.

Lemma 3.2.6. *Fix a key P-witness w for a congruence $\sim$ on Q and a key aide w'. If w is a maximal P-witness, then every mesoprimary decomposition $\bigcap_i \sim_i$ for $\sim$ has a component with either w or w' as a cogenerator.*

Proof. Suppose w is maximal among P-witnesses for $\sim$. Upon localizing at P, any component whose associated prime is not contained in P becomes the total congruence, and the remaining components decompose $\sim$ on Q_P. Thus, it suffices to replace Q with Q_P, so that P is maximal. If w' is nil, then w is a cogenerator for $\sim$, so it is a cogenerator for any P-primary component $\sim_i$ under which it is not nil. If w' lies in the same Green's class as w in Q_P, then we are done by Lemma 3.2.4. Now, assume w' is not nil and lies in a different Green's class in Q_P. Since $w \not\sim w'$, some component $\sim_i$ separates w and w'. Localization Q at any prime P' properly contained in P identifies w and w' since $w + p = w' + p$ for any $p \in P \setminus P'$. This means any P'-primary component also identifies w and w', so $\sim_i$ must be P-primary. Since w is maximal among P-witnesses, either it is a cogenerator for $\sim_i$, or it is nil modulo $\sim_i$; the latter implies that w' is a cogenerator for $\sim_i$. In either case, the proof is complete. □

We now present the first main result of this section.

Theorem 3.2.7. *Fix a true witness w for a prime $P \subset Q$, and let $\approx$ denote the P-prime congruence at w.*

1. *If (i) the discrete testimony of w is not suspicious, or (ii) $\approx$ is the P-prime congruence at some non-nil key aide w' for w, then $\approx$ appears as the associated prime congruence of some mesoprimary component in each mesoprimary decomposition $\bigcap_i \sim_i$ of $\sim$.*

2. *If w satisfies neither (i) nor (ii), then the component in the coprincipal decomposition in Theorem 3.1.6 with cogenerator w is redundant.*

Proof. If the discrete testimony of w is not suspicious, then apply Lemma 3.2.4. On the other hand, if w has a key aide w' whose prime congruence is also $\approx$, then by

Lemma 3.2.6 one of w and w' must appear as a cogenerator of some component $\sim_i$. This proves the first statement.

Now, fix $a, b \in Q$ with $a \not\sim b$. By Corollary 2.2.16, there exists a prime $P \subset Q$ and $u \in Q$ such that, after possibly swapping a and b, $a + u$ is a key witness with key aide $b + u$. If $a + u$ has suspicious discrete testimony, then by Proposition 3.1.3 it does not have nil as a key aide, so $b + u$ is also a key witness for P. If, additionally, $a + u$ and $b + u$ have distinct P-prime congruences, then since $a + u$ and $b + u$ have identical discrete testimony, the discrete testimony of $b + u$ is not suspicious. Since $a \not\sim_{b+u}^{P} b$, this proves the second statement. $\square$

Corollary 3.2.8. *For each congruence $\sim$, there exists a set $A(\sim)$ of prime congruences such that each congruence in $A(\sim)$ appears in every mesoprimary decomposition of $\sim$, and there exists a mesoprimary decomposition of $\sim$ in which the associated prime congruence of each component lies in $A(\sim)$.*

Theorem 3.2.9. *Fix a mesoprimary decomposition $\bigcap_i \sim_i$ of a congruence $\sim$ on Q. If P is a minimal associated prime of $\sim$, then every true P-witness w of $\sim$ appears as a cogenerator of some component $\sim_i$.*

Proof. Let $\approx$ denote the P-prime congruence at w, and let $\approx_i$ denote the prime congruence associated to $\sim_i$ for each i. If $P = \varnothing$, then since P is associated to $\sim$, some component $\sim_i$ is P-primary, and in fact $\sim_i = \approx$. Now assume P is nonempty. Once again, after localizing at P, assume P is maximal. Since P is a minimal associated prime, $\sim$ is P-primary by Theorem 2.2.8. Since w is true, either it is a maximal P-witness, in which case it has ∞ as a key aide, or its testimony is not suspicious. In either case, we are done by Lemma 3.2.4. $\square$

We now present two minimality results for congruences.

Corollary 3.2.10. *Any congruence $\sim$ on Q with no embedded associated monoid primes has a unique irredundant induced coprincipal decomposition. In particular, the induced coprincipal decomposition of $\sim$ with one component at each true witness is irredundant.*

Proof. The common refinement of the coprincipal components of $\sim$ cogenerated by true witnesses is a coprincipal decomposition by Theorem 3.1.6. Moreover, excluding a component cogenerated by a true witness $w \in Q$ from this decomposition (or any induced coprincipal decomposition) results in a mesoprimary decomposition lacking a component with cogenerator w, violating Theorem 3.2.9. □

Corollary 3.2.11. *Any congruence $\sim$ on Q with no embedded associated monoid primes has a unique minimal induced mesoprimary decomposition. In particular, upon replacing any set of components from the decomposition in Corollary 3.2.10 that share an associated prime congruence with their common refinement, the resulting mesoprimary decomposition is minimal with one component per truly associated prime congruence.*

Proof. Each component of the resulting decomposition is mesoprimary by (Kahle and Miller, 2013, Proposition 6.9), and the decomposition is mesoprimary by Corollary 3.2.10. Moreover, this decomposition is minimal since no two components share an associated prime congruence. □

3.3 Irredundant mesoprimary decompositions of binomial ideals

This section contains minimality results for mesoprimary decompositions of binomial ideals analogous to those for congruences in Section 3.2. The following definition appeared as (Kahle and Miller, 2013, Definition 17.8).

Definition 3.3.1. A mesoprime ideal $I_{\rho,P}$ is *truly associated* to a binomial ideal I if it is the P-mesoprime at some character I-witness $\mathbf{x}^w$.

Remark 3.3.2. The notion of truly associated mesoprime differs from that of truly associated prime congruence because the intersection of the mesoprimes in the testimony of an I-witness may not be a binomial ideal; see (Kahle and Miller, 2013, Example 16.6). Theorem 3.3.3 implies that this happens exactly when a non-true witness is a character witness.

Theorem 3.3.3. *Fix a binomial ideal $I \subset \Bbbk[Q]$. If $w \in Q$ is a character I-witness that is not a true witness for $\sim_I$, then the intersection of the mesoprimes in its testimony is not binomial.*

Proof. Let J denote the intersection of the mesoprimes in $T_P(w)$, and

$$J' = \langle \mathbf{x}^a - \lambda \mathbf{x}^b \in J : a, b \in Q, \lambda \in \Bbbk \rangle \subset J$$

denote the largest binomial ideal contained in J. Since J contains I_w^P, so does J', and since w it not a true witness for $\sim_I$, J' and I_w^P induce the same congruence on Q. Additionally, I_w^P is maximal among binomial ideals inducing its congruence on Q, so $I_w^P = J'$. Since w is a character witness, J properly contains I_w^P and thus J', so J is not binomial. $\square$

Definition 3.3.4. A mesoprimary decomposition $I = \bigcap_i I_i \subset \Bbbk[Q]$ is

1. *minimal* if I_i and I_j have distinct associated mesoprime ideals for $i \neq j$;
2. *irredundant* if no I_i can be omitted;
3. *coprincipal* if each I_i is coprincipal;
4. *induced* if each I_i is an intersection of coprincipal components cogenerated by witnesses for the same associated mesoprime.

Example 3.3.5. As in Section 3.2, we restrict our attention to induced mesoprimary decompositions, for the same reasons given in Remark 3.2.2. In particular, the non-true witness for the ideal given in Example 3.2.3 is also not a character witness.

The following lemma first appeared as (Kahle and Miller, 2013, Lemma 13.4).

Lemma 3.3.6. *If $I \subset \Bbbk[Q]$ is a P-mesoprimary ideal, then monomial localization along a monoid prime is either injective or 0 on $\Bbbk[Q]/I$, with injectivity precisely when the prime contains P.*

Here is the binomial analog of Lemma 3.2.4.

Proposition 3.3.7. *Fix a mesoprimary decomposition $\bigcap_i I_i$ for a binomial ideal $I \subset \Bbbk[Q]$, and fix an I-witness $w \in Q$ with associated monoid prime P. If the testimony of w is not suspicious, then $\mathbf{x}^w$ cogenerates some component I_i.*

Proof. By Lemma 3.3.6, it suffices to work in the monomial localization along P. By assumption, the intersection of the mesoprimes in the testimony $T_P(w)$ of w properly contains the P-mesoprime I_w^P at w, so some $f \notin I_w^P$ lies in this intersection. This means that $\mathbf{x}^w f$ lies outside of I. It follows that $\mathbf{x}^w f$ lie outside of some component I_j. Since each mesoprime in $T_P(w)$ is a P-mesoprime, we can assume the exponent of each monomial in f lies in $Q \backslash P$. Any component I_i with a cogenerator of the form $\mathbf{x}^{w+p}$ for some $p \in P$ contains $\mathbf{x}^w f$ since $f \in I_{w+p}^P$, so I_j has no cogenerators of this form. However, $\mathbf{x}^w f$ is nonzero modulo I_j, so $\mathbf{x}^w$ is nonzero modulo I_j. This means $\mathbf{x}^w$ cogenerates I_j. □

Theorem 3.3.8. *Fix a mesoprimary decomposition $\bigcap_i I_i$ for a binomial ideal $I \subset \Bbbk[Q]$, and a character I-witness $w \in Q$ for P. If P is minimal among monoid primes associated to I, then w cogenerates some component I_i.*

Proof. By Lemma 3.3.6, it suffices to work in the monomial localization along P. Since P is minimal among monoid primes associated to I, I_P induces a P-primary congruence. If w is maximal among I-witnesses for P, then $\mathbf{x}^w$ is a cogenerator for I, meaning it must occur as a cogenerator of any component under which it is nonzero.

If w is not maximal, then the testimony $T_P(w)$ of w is not suspicious, so we are done by Lemma 3.3.7. $\square$

Corollaries 3.3.9 and 3.3.10 are direct binomial analogues of Corollaries 3.2.10 and 3.2.11, respectively. Their proofs also follow from Theorems 3.3.8 and 2.3.9 in a manner similar to their combinatorial analogues and thus are omitted.

Corollary 3.3.9. *Any binomial ideal $I \subset \Bbbk[Q]$ with no embedded associated monoid primes has a unique irredundant induced coprincipal decomposition. In particular, the mesoprimary decomposition of I given in Theorem 2.3.9 with one component cogenerated by each character witness is irredundant.*

Corollary 3.3.10. *Any binomial ideal $I \subset \Bbbk[Q]$ with no embedded associated monoid primes has a unique minimal induced mesoprimary decomposition. In particular, upon replacing any components in the decomposition in Corollary 3.3.9 that share an associated mesoprime with their intersection, the resulting mesoprimary decomposition is minimal with one component per truly associated mesoprime.*

4

Irreducible decomposition of binomial ideals

An ideal is irreducible if it cannot be written as an intersection of two ideals properly containing it. It is easy to show that irreducible ideals are primary, and that any ideal I in a Noetherian ring can be written as an intersection of irreducible ideals. These *irreducible decompositions* are thus a special case of primary decompositions, but likewise are hard to compute in general. If I is a monomial ideal, however, this task is much easier. In particular, any monomial ideal can be written as an intersection of irreducible ideals that are themselves monomial ideals, and these *monomial irreducible decompositions* are heavily goverend by combinatorics. This, together with the results from Chapter 1, motivates the following question, which first appeared as (Eisenbud and Sturmfels, 1996, Problem 7.3).

Question A. Do binomial ideals over an algebraically closed field admit binomial irreducible decompositions?

In this chapter, we investigate Question A using the theory of mesoprimary decomposition. We introduce soccular congruences as combinatorial approximations of irreducible binomial quotients, and we construct soccular decompositions of monoid congruences by coarsening the coprincipal components used to construct mesoprimary decompositions in Theorem 2.2.15. Next, we introduce binoccular ideals (Definition 4.3.2), which are the binomial analogues of soccular congruences, and similarly construct binoccular decompositions of binomial ideals by coarsening the coprincipal components in Definition 2.3.5. Finally, we produce (not necessarily binomial) irreducible decompositions for any binomial ideal (Corollary 4.5.7), and demonstrate in Example 4.4.1 a binomial ideal that cannot be written as an intersection of binomial irreducible ideals, thus answering Question A in the negative.

4.1 Soccular congruences

Lemma 4.1.2 requires a definition not used thus far.

Definition 4.1.1. Fix an ideal I in a Noetherian ring R and a prime ideal $\mathfrak{p} \subset R$. The $\mathfrak{p}$-*socle of* I is given by

$$\operatorname{soc}_{\mathfrak{p}}(I) = \{f \in R_{\mathfrak{p}}/I_{\mathfrak{p}} : \mathfrak{p}f = 0\} \subseteq R_{\mathfrak{p}}/I_{\mathfrak{p}}.$$

The socle of I is *simple* if $\dim_{\Bbbk(\mathfrak{p})}(\operatorname{soc}_{\mathfrak{p}}(I)) = 1$, where $\Bbbk(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ denotes the residue field at $\mathfrak{p}$.

Lemma 4.1.2. *The number of components in any irredundant irreducible decomposition of a $\mathfrak{p}$-primary ideal I in a Noetherian ring R equals* $\dim_{\Bbbk(\mathfrak{p})} \operatorname{soc}_{\mathfrak{p}}(I)$.

Proof. (Vasconcelos, 1998, Proposition 3.15). □

Although the equivalent conditions in Theorem 2.2.10 for mesoprimary quotients are strong, they do not imply that a binomial inducing a coprincipal congruence has a simple socle. Example 4.1.3 demonstrates a primary coprincipal binomial ideal that is not irreducible.

Example 4.1.3. The congruence on $\mathbb{N}^2$ induced by the ideal $I = \langle x^2 - xy, xy - y^2, x^3 \rangle$ from Example 2.1.5 is coprincipal, but $x - y \in \mathrm{soc}_{\mathfrak{m}}(I)$ for $\mathfrak{m} = \langle x, y \rangle$. This is because x and y are both key witnesses and each is an aide for the other.

With this in mind, we introduce soccular congruences.

Definition 4.1.4. A congruence $\sim$ on Q is *soccular* if each of its key witnesses generates the same ideal in the localized quotient $Q_P/\sim$.

Definition 4.1.5 demonstrates how to methodically remove this behavoir from a given coprincipal congruence. The construction of soccular decomposition in Theorem 4.2.2 relies on this method.

Definition 4.1.5. Fix a P-coprincipal congruence $\sim$ on Q with cogenerator w. The *(1st) soccular collapse of* $\sim$ is the congruence $\approx$ that sets $a \approx b$ if $a, b \notin \langle w \rangle$ and $a + p \sim b + p$ for all $p \in P$. The *i-th soccular collapse of* $\sim$ is the soccular collapse of the $(i-1)$-st soccular collapse of $\sim$.

In general, to form a congruence from a given set of relations, one takes monoid closure and then transitive closure. Lemma 4.1.6 says that for a soccular collapse of a coprincipal congruence, both of these operations are trivial.

Lemma 4.1.6. *The soccular collapse of a P-coprincipal congruence $\sim$ is a congruence on Q that coarsens $\sim$.*

Proof. The soccular collapse $\approx$ is symmetric and transitive since $\sim$ is symmetric and transitive. Suppose $a, b \notin \langle w \rangle$ with $a + p \sim b + p$ for all $p \in P$. Then for all $q \in Q$, $a + q + p \sim b + q + p$ for all $p \in P$ since $q + p \in P$, so $a + q \approx b + q$. Therefore $\approx$ is a congruence on Q. Lastly, whenever $a \sim b$ we clearly have $a + p \sim b + p$ for all $p \in P$, so $\sim$ refines $\approx$. $\square$

The following is a useful consequence of Lemma 4.1.6.

Lemma 4.1.7. *Resuming the notation from Definition 4.1.5, if $a \approx b$ and $a \not\sim b$, then neither a nor b is maximal in Q modulo Green's relation.*

Proof. Given Lemma 4.1.6, the definition of $\approx$ ensures that a and b both preceed w modulo Green's relation, which ensures a and b are not maximal. □

Lemma 4.1.8 shows that taking the soccular collapse of a coprincipal congruence does not modify Green's classes.

Lemma 4.1.8. *Resuming the notation from Definition 4.1.5, if $a, b \in Q$ differ by a cancellative modulo $\sim$, then soccular collapse does not join them.*

Proof. Suppose $a \approx b$ and $a = b + f$ for some cancellative element f. For each $p \in P$, $a + p = b + p$ by Lemma 4.1.6, and each is non nil by Lemma 4.1.7. Thus $b + f + p \sim b + p$, so $f = 0$ by partial cancellativity of $b + p$. □

We now summarize these results in Proposition 4.1.9 below.

Proposition 4.1.9. *Fix a P-coprincipal congruence $\sim$ on Q with cogenerator w. The soccular collapse $\approx$ of $\sim$ is coprincipal with cogenerator w, and $\approx$ coarsens $\sim$. Moreover, the elements $a, b \in Q$ distinct under $\sim$ but identified under $\approx$ are precisely the key witnesses of $\sim$ lying outside the Green's class of w.*

Proof. The congruence $\approx$ coarsens $\sim$ by Lemma 4.1.6. Since $\sim$ is mesoprimary, Lemma 4.1.8 ensures that the action of $F = Q \backslash P$ on $Q/\!\approx$ satisfies Theorem 2.2.10, and by Lemma 4.1.7, $\approx$ agrees with $\sim$ on the Green's class of w. The final claim follows upon observing that a and b are by definition key witnesses for $\sim$. □

Definition 4.1.10. Fix a P-coprincipal congruence $\sim$ on Q. Two distinct key witnesses $a, b \in Q$ for $\sim$ form a *key witness pair* if each is a key aide for the other.

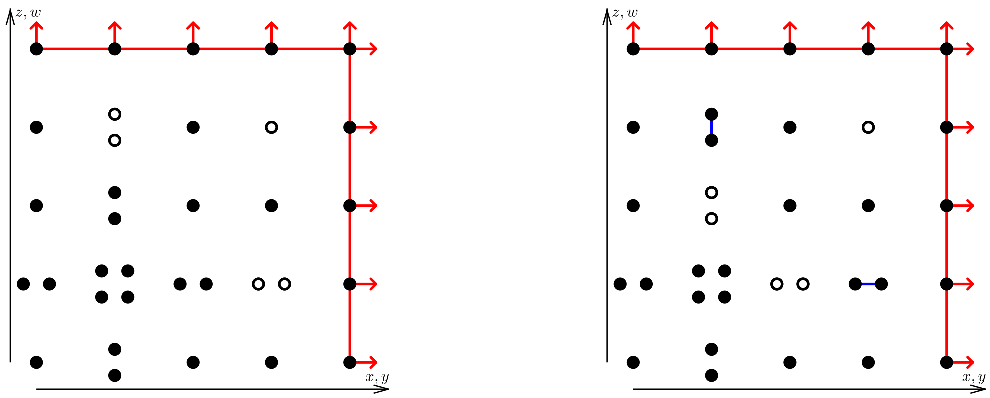


FIGURE 4.1: The congruence induced by $\langle x^2-xy, xy-y^2, x^3, z^2-zw, zw-w^2, z^3\rangle \subset \Bbbk[x,y,z,w]$ on $\mathbb{N}^4$ (left), together with its soccular collapse (right). The congruence $\sim_I$ has five key witnesses (marked on the left with white dots) and four protected witnesses (marked on the right with white dots).

Remark 4.1.11. Any key witness pair under a coprincipal congruence $\sim$ neither of which is Green's equivalent to the cogenerator w is no longer a key witness pair under the soccular collapse $\approx$ of $\sim$ by Proposition 4.1.9. However, $\approx$ may still have key witnesses, as shown in Examples 4.1.12 and 4.1.13.

Example 4.1.12. Let $I = \langle x^2-xy, xy-y^2, x^3, z^2-zw, zw-w^2, z^3\rangle \subset \Bbbk[x,y,z,w]$. The monoid $\mathbb{N}^4/{\sim_I}$, shown on the left in Figure 4.1, is isomorphic to the cartesian square of the quotient monoid in Example 4.1.3. The quotient of $\mathbb{N}^4$ by the soccular collapse $\approx$ of $\sim_I$ is shown on the right in Figure 4.1. The key witnesses in each congruence are marked with white dots. Notice that x^2z and x^2w are key witnesses for $\approx$ but not for $\sim$.

Example 4.1.13. Let $I = \langle x^3-x^2y, x^2y-xy^2, xy^3-y^4, x^5\rangle \subset \Bbbk[x,y]$. The congruence $\sim_I$ and its soccular collapse are shown in Figure 4.2. The monoid element xy is a key witness for $\sim_I$, where it is paired with y^2, as well as for the key witness coarsening of $\sim_I$, where it is paired with x^2.

The previous two examples motivate Definition 4.1.14.

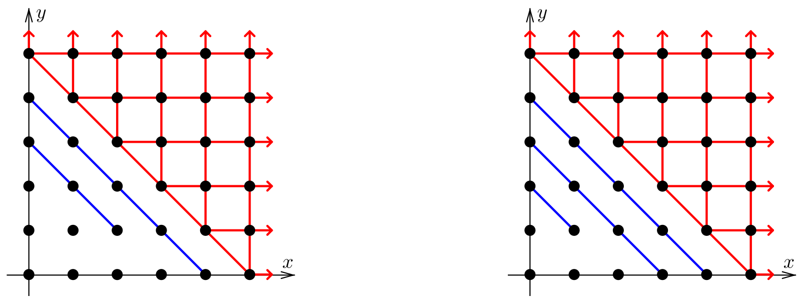


FIGURE 4.2: The congruence induced by $I = \langle x^3 - x^2y, x^2y - xy^2, xy^3 - y^4, x^5\rangle \subset \Bbbk[x,y]$ on $\mathbb{N}^2$ (left), together with its soccular collapse (right). The monomial xy is a key witness for both $\sim_I$ and its key witness coarsening.

Definition 4.1.14. Fix a coprincipal congruence $\sim$ on Q with cogenerator w. An element $a \in Q$ is a *protected witness for* $\sim$ if it is a key witness for the ith soccular collapse of $\sim$ for some $i \geqslant 1$. Elements $a, b \in Q$ form a *protected witness pair* if they form a key witness pair for some iterated soccular collapse of $\sim$.

Lemma 4.1.15 follows from the fact that Q is a Noetherian monoid.

Lemma 4.1.15. *Fix a coprincipal congruence $\sim$ on Q, and let $\sim_i$ denote the i-th soccular collapse of $\sim$. Then $\sim_i = \sim_{i+1}$ for $i \gg 0$.* □

Definition 4.1.16. Fix a coprincipal congruence $\sim$ on Q. The *soccular closure* $\overline{\sim}$ of $\sim$ is the congruence refined by $\sim$ that additionally relates any a and b related under some soccular collapse of $\sim$.

Lemma 4.1.17. *Fix a coprincipal congruence $\sim$ on Q with cogenerator w. The soccular closure $\overline{\sim}$ of $\sim$ is a soccular congruence, and its set of key witnesses is exactly the Green's class of w.*

Proof. By construction, the soccular closure has no key witnesses outside the Green's class of w. □

We now characterize protected witnesses. In what follows, let

$$(w :_\sim q) = \{p \in Q : \overline{q} + \overline{p} = \overline{w} \text{ in } Q_P/{\sim}\}.$$

Theorem 4.1.18. *Fix a P-coprincipal congruence $\sim$ on Q with cogenerator w, and write $\overline{Q} = Q/{\sim}$. Then $q, q' \in Q$ with distinct classes in $\overline{Q}$ are protected witnesses for $\sim$ related under some soccular collapse of $\sim$ if and only if $(w :_\sim q) = (w :_\sim q')$.*

Proof. Let $\overline{\sim}$ denote the soccular closure of $\sim$. Since w has the same equivalence class under $\sim$ and $\overline{\sim}$, $(w :_\sim q) = (w :_{\overline{\sim}} q)$ for all $q \in Q$. Thus, if q and q' are distinct under $\sim$ but merged under $\overline{\sim}$, then $(w :_\sim q)$ and $(w :_\sim q')$ must coincide.

Now, assume q and q' are not related under $\overline{\sim}$. Let $p \in P$ be maximal such that $q + p$ and $q' + p$ are distinct under $\approx$, so that $q + p$ and $q' + p$ become merged under the action of any element of P. Since $\approx$ has no key witness pairs, one of $q + p$ and $q' + p$ must be nil, and maximality of p implies the other is Green's equivalent to w. After possibly switching q and q', this gives $p \in (w :_\sim q)$ but $p \notin (w :_\sim q')$. □

A non-iterative characterization of soccular closure follows from Theorem 4.1.18.

Corollary 4.1.19. *Fix a coprincipal congruence $\sim$ on Q cogenerated by w. The soccular closure $\overline{\sim}$ of $\sim$ relates a and b if and only if $(w :_\sim a) = (w :_\sim b)$.* □

4.2 Soccular decompositions of congruences

In this section, we give a constructive proof that every congruence can be expressed as a common refinement of soccular congruences. We first see this in Corollary 4.2.3, though the resulting decomposition is not a mesoprimary decomposition (see Remark 4.2.4). We then remove unnecessary components from this decomposition in Theorem 4.2.5, and show that the resulting decomposition is mesoprimary.

Definition 4.2.1. Fix a P-coprincipal congruence $\sim$ on Q and a key witness $w \in Q$. The *soccular component $\overline{\sim}_w^P$ of $\sim$ cogenerated by w along P* is the soccular closure of the coprincipal component $\sim_w^P$ cogenerated by w along P.

We are now ready to prove the first main result regarding soccular decomposition of congruences.

Theorem 4.2.2. *Any coprincipal congruence $\sim$ on Q is the common refinement of the soccular components cogenerated by its protected witnesses.*

Proof. Each soccular component coarsens $\sim$ by Lemma 4.1.6, so it suffces to show that their common refinement is $\sim$. Let $w \in Q$ denote a cogenerator of $\sim$, and fix distinct $a, b \in Q$. If the soccular component of $\sim$ at w (that is, the soccular closure of $\sim$) leaves a and b distinct, we are done. Otherwise, both a and b are protected witnesses, and the soccular component of $\sim$ at a joins b with the nil class. $\square$

Corollary 4.2.3. *Any congruence $\sim$ on Q can be expressed as a common refinement of soccular congruences.*

Proof. Apply Theorem 2.2.15 to $\sim$, then Theorem 4.2.2 to each component. $\square$

Remark 4.2.4. The proof of Corollary 4.2.3 shows that any congruence $\sim$ on Q admits a soccular decomposition by first finding a coprincipal decomposition with one component per key witness, courtesy of Theorem 2.2.15, then further refining each coprincipal component into soccular components with one component per protected witness. It should be noted that this decomposition is not necessarily a mesoprimary decomposition in the sense of Definition 2.2.14, since the associated prime congruence of a component $\approx$ cogenerated at a protected witness $q \in Q$ need not coincide with the prime congruence at q under $\sim$. Theorem 4.2.5, on the other hand, shows that the components in this decomposition cogenerated at non-key protected witnesses are redundant, and the resulting decomposition is indeed a mesoprimary decomposition.

Theorem 4.2.5. *Any congruence $\sim$ is the common refinement of the soccular components cogenerated by its key witnesses.*

Proof. For elements $a, b \in Q$ with $a \not\sim b$, Corollary 2.2.16 produces, after possibly swapping a and b, a prime $P \subset Q$ and $u \in Q$ such that $a \not\sim_w^P b$ for a key witness $w = a + u$ with key aide $b + u$. Since $\approx_w^P$ has the same cogenerator and nil class as $\sim_w^P$, Corollary 4.1.19 ensures that $\approx_w^P$ does not relate a and b as well. □

4.3 Soccular decompositions of binomial ideals

In this section, we present the analogue for binomial ideals of Theorem 4.2.5. As with coprincipal decomposition, non-key witnesses are required.

Definition 4.3.1. Fix a binomial ideal $I \subset \Bbbk[Q]$ and a prime monoid ideal $P \subset Q$. The *P-socle of I* is the ideal

$$\mathrm{soc}_P(I) = \{f \in \Bbbk[Q]_P/I_P : \mathfrak{m}_P f = 0\} \subset \Bbbk[Q]_P/I_P.$$

Definition 4.3.2. A binomial ideal $I \subset \Bbbk[Q]$ is *binoccular* if it is P-coprincipal and every nonzero binomial in $\mathrm{soc}_P(I)$ is a monomial cogenerator of $\Bbbk[Q]_P/I_P$.

Example 4.3.3. Binoccular ideals need not induce soccular congruences. Let $I = \langle x^2 - xy, xy + y^2, x^3\rangle$. The monomials x and y form a key witness pair for $\sim_I$, but I is irreducible, so these monomials do not form a binomial socle element.

Definition 4.3.4 and Proposition 4.3.6 are the binomial analogues of Definition 4.1.5 and Proposition 4.1.9, respectively.

Definition 4.3.4. Fix a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ cogenerated by $w \in Q$. The *(1st) binoccular collapse of I* is the ideal

$$I_1 = \langle \mathbf{x}^a - \lambda\mathbf{x}^b \mid \mathbf{x}^p(\mathbf{x}^a - \lambda\mathbf{x}^b) \in I \text{ for all } p \in P\rangle$$

and the *i-th binoccular collapse I_i of I* is the binoccular collapse of I_{i-1}. The *binoccular closure of I* is the smallest ideal containing all binoccular collapses of I.

Lemma 4.3.5 follows from the fact that $\Bbbk[Q]$ is Noetherian.

Lemma 4.3.5. *Fix a coprincipal binomial ideal I in $\Bbbk[Q]$, and let I_i denote the i-th binoccular collapse of I. Then $I_i = I_{i+1}$ for $i \gg 0$.* $\square$

Proposition 4.3.6. *Fix a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ cogenerated by $w \in Q$. The binoccular collapse J of I is also a coprincipal ideal cogenerated by w, and for any binomial $\mathbf{x}^a - \lambda\mathbf{x}^b \in J$ that lies outside of I, the elements a and b form a key witness pair for $\sim_I$.*

Proof. This follows from Definition 4.3.4 and Proposition 4.1.9 after noticing that the congruence $\sim_J$ coarsens $\sim_I$ and refines $\overline{\sim}_I$. $\square$

Binoccular components comprise the decomposition in Theorem 4.3.10.

Definition 4.3.7. Fix a binomial ideal $I \subset \Bbbk[Q]$, a prime $P \subset Q$, and $w \in Q$. The *binoccular component of I cogenerated by w* is the binoccular closure $\overline{W}_w^P(I)$ of the coprincipal component $W_w^P(I)$ of I cogenerated by w along P.

Lemma 4.3.8 is the core of the original proof by Kahle and Miller of Theorem 2.3.9, but it was not stated explicitly in these terms. This unifying principle is also important as we construct binoccular decompositions of binomial ideals (Theorem 4.3.10) and irreducible decompositions of binomial ideals (Theorem 4.5.6).

Lemma 4.3.8. *Fix a binomial ideal $I \subset R = \Bbbk[Q]$ and (not necessarily binomial) ideals $W_1, \ldots, W_r$ containing I. The following are equivalent.*

1. *$I = W_1 \cap \cdots \cap W_r$.*
2. *The natural map $R/I \to R/W_1 \oplus \cdots \oplus R/W_r$ is injective.*
3. *The natural map $\operatorname{soc}_P(I) \to R_P/(W_1)_P \oplus \cdots \oplus R_P/(W_r)_P$ is injective for every monoid prime $P \subset Q$ associated to $\sim_I$.*

4. *The natural map* $\mathrm{soc}_{\mathfrak{p}}(I) \to R_{\mathfrak{p}}/(W_1)_{\mathfrak{p}} \oplus \cdots \oplus R_{\mathfrak{p}}/(W_r)_{\mathfrak{p}}$ *is injective for every prime* $\mathfrak{p} \in \mathrm{Ass}(I)$.

Proof. The containments $I \subseteq W_1, \ldots, I \subseteq W_r$ induce a well defined homomorphism

$$R/I \to R/W_1 \oplus \cdots \oplus R/W_r.$$

whose kernel is exactly $W_1 \cap \cdots \cap W_r$ modulo I. We have $I = W_1 \cap \cdots \cap W_r$ if and only if this map is injective; therefore $1 \Leftrightarrow 2$.

Assume the homomorphism just constructed is injective. Exactness of localization produces an injective map

$$R_P/I_P \to R_P/(W_1)_P \oplus \cdots \oplus R_P/(W_r)_P$$

for each monoid prime $P \subset Q$. This proves $2 \Rightarrow 3$.

Now assume statement 3 holds, and fix a prime $\mathfrak{p} \in \mathrm{Ass}(I)$. By (Kahle and Miller, 2013, Lemma 15.2), $\mathfrak{p}$ is associated to some mesoprime $I_{\rho,P}$ associated to I. Since P is associated to $\sim_I$, the map

$$\mathrm{soc}_P(I) \to R_P/(W_1)_P \oplus \cdots \oplus R_P/(W_r)_P$$

is injective. Every monomial outside of $\mathfrak{m}_P$ also lies outside of $\mathfrak{p}$, so by inverting the remaining elements outside of $\mathfrak{p}$, we obtain the injection

$$\mathrm{soc}_P(I)_{\mathfrak{p}} \to R_{\mathfrak{p}}/(W_1)_{\mathfrak{p}} \oplus \cdots \oplus R_{\mathfrak{p}}/(W_r)_{\mathfrak{p}}.$$

Any element in $\mathrm{soc}_P(I)_{\mathfrak{p}}$ is annihilated by $\mathfrak{m}_P$, so $\mathrm{soc}_{\mathfrak{p}}(I) \subset \mathrm{soc}_P(I)_{\mathfrak{p}}$, yielding $3 \Rightarrow 4$.

Finally, suppose statement 4 holds, and fix a nonzero $f \in R/I$ and a prime $\mathfrak{p}$ minimal over the annihilator of f. The image $\overline{f} \in R_{\mathfrak{p}}/I_{\mathfrak{p}}$ of f is nonzero since $\mathfrak{p}$ contains the annihilator of f. Minimality of $\mathfrak{p}$ implies some power of $\mathfrak{p}$ annihilates $\overline{f}$, so $a\overline{f}$ is annihilated by $\mathfrak{p}$ for some $a \in \mathfrak{p}$. By assumption, $a\overline{f}$ has nonzero image in some $(R/W_i)_{\mathfrak{p}}$, meaning af has nonzero image in R/W_i. This proves $4 \Rightarrow 2$. $\square$

Lemma 4.3.9. *Fix a binomial ideal $I \subset \Bbbk[Q]$ and a monoid prime $P \subset Q$. Each $f \in \mathrm{soc}_P(I)$ has a nonzero monomial that is an I_P-witness for P.*

Proof. Fix $f \in \mathrm{soc}_P(I)$. Fix a nonzero monomial $\lambda\mathbf{x}^w$ of f such that w is minimal among the monomials of f under Green's preorder on Q. For each monomial $\mathbf{x}^p \in \mathfrak{m}_P$, we have $\mathbf{x}^p f = 0$, so $\lambda\mathbf{x}^w$ must either cancel with some other monomial in f or become zero; the latter case implies that $w + p \sim_I \infty$. Either way, w becomes non-singleton under the kernel of each cover morphism in P. Minimality of w ensures that w is not exclusively maximal, meaning that w is an I_P-witness for P. □

Theorem 4.3.10 implies that every binomial ideal can be written as an intersection of binoccular binomial ideals.

Theorem 4.3.10. *For any binomial ideal $I \subset \Bbbk[Q]$, the intersection of the binoccular components cogenerated by the I-witnesses constitutes a combinatorial mesoprimary decomposition of I.*

Proof. Fix a monoid prime $P \subset Q$ associated to $\sim_I$ and a nonzero $f \in \mathrm{soc}_P(I)$. By Lemma 4.3.8, it suffices to show that f is nonzero modulo the localization along P of some binoccular component. By Lemma 4.3.9, some nonzero monomial $\lambda\mathbf{x}^w$ of f is an I_P-witness for P. This means every monomial of f other than $\lambda\mathbf{x}^w$ that is nonzero modulo $W_w^P(I)_P$ is Green's equivalent to w, so f has nonzero image in $\overline{W}_w^P(I)_P$. □

4.4 Nonexistence of binomial irreducible decomposition

The socle of a binoccular binomial ideal has exactly one binomial, namely its monomial cogenerator. However, its socle may still contain non-binomial elements, as Example 4.4.1 demonstrates. Theorem 4.4.2 shows that the ideal in Example 4.4.1 cannot be written as the intersection of irreducible binomial ideals, answering Question A in the negative.

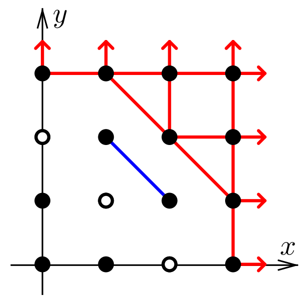


FIGURE 4.3: The congruence induced by $\langle x^2y - xy^2, x^3, y^3\rangle \subset \Bbbk[x,y]$ on $\mathbb{N}^2$. The non-binomial element $x^2 + y^2 - xy$ lies in the socle of I, and as such, I does not admit a binomial irreducible decomposition.

Example 4.4.1. Let $I = \langle x^2y - xy^2, x^3, y^3\rangle \subset \Bbbk[x,y]$. Its congruence is depicted in Figure 4.3. The binomial relation forces $x^2y^2 \in I$, so I is cogenerated by x^2y. The monomials x^2, xy and y^2 are all non-key witnesses, and $x^2 + y^2 - xy \in \mathrm{soc}_P(I)$ for $\mathfrak{m}_P = \langle x, y\rangle$. The expression $I = \langle x^2 + y^2 - xy, x^3, y^3\rangle \cap \langle x^3, y\rangle$ is an irreducible decomposition of I, and by Theorem 4.4.2, every irreducible decomposition of I contains some non-binomial irreducible component.

Theorem 4.4.2. *The ideal $I = \langle x^2y - xy^2, x^3, y^3\rangle \subset \Bbbk[x,y]$ cannot be expressed as an intersection of irreducible binomial ideals.*

Proof. Let $\mathfrak{m}_P = \langle x, y\rangle$. The $\Bbbk$-vector space $\mathrm{soc}_P(I)$ is spanned by $\alpha = x^2 + y^2 - xy$ and $\beta = x^2y$. Since $\dim_\Bbbk(\mathrm{soc}_P(I)) = 2$, any irredundant irreducible decomposition of I has exactly 2 components. Suppose $I = I_1 \cap I_2$ with I_1, I_2 irreducible. By Lemma 4.3.8, this means $\alpha + \lambda\beta \in I_i$ for some $\lambda \in \Bbbk$ and $i \in \{1, 2\}$. But $I + \langle \alpha + \lambda\beta\rangle$ is already irreducible, so $I_i = I + \langle \alpha + \lambda\beta\rangle$. $\square$

Example 4.4.1 is the first example of a binomial ideal that does not admit a binomial irreducible decomposition. However, it is still possible to construct a (not neccesarily binomial) irreducible decomposition from essentially combinatorial data, as Corollary 4.5.7 demonstrates.

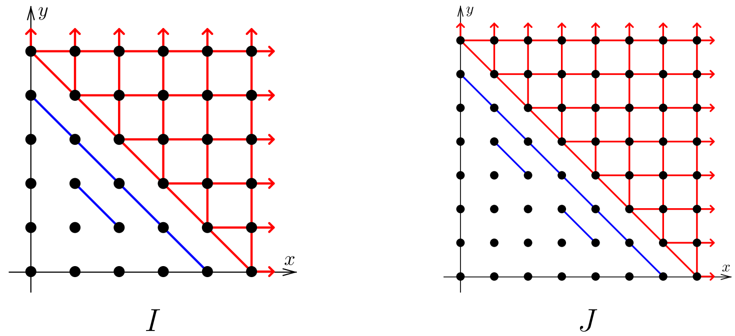


FIGURE 4.4: The congruences induced by $I = \langle x^2y - xy^2, x^4 - x^3y, xy^3 - y^4, x^5\rangle$ (left) and $J = \langle x^4y - x^3y^2, x^2y^3 - xy^4, x^6 - x^5y, xy^5 - y^6, x^7\rangle$ (right) on $\mathbb{N}^2$. I admits a binomial irreducible decomposition, but J does not.

Example 4.4.3 exhibits the difficulties in determining whether or not a given binomial ideal admits a binomial irreducible decomposition. This question is thus closely connected with understanding which components in a coprincipal decomposition are redundant; see Problem 6.3.

Example 4.4.3. Consider the ideals $I = \langle x^2y - xy^2, x^4 - x^3y, xy^3 - y^4, x^5\rangle$ and $J = \langle x^4y - x^3y^2, x^2y^3 - xy^4, x^6 - x^5y, xy^5 - y^6, x^7\rangle$, whose respective congruences are depicted in Figure 4.4. The ideal I has 3 key witnesses aside from the cogenerator, and the binoccular decomposition produced in Theorem 4.3.10 has a component at each of these key witnesses. Any one of these 3 can be omitted, and omitting the component cogenerated by x^2y yields a binomial irreducible decomposition of I. However, J has 4 non-maximal key witnesses, two of which fail to cogenerate binoccular components that admit binomial irreducible decompositions. Since only one can be omitted, J does not admit a binomial irreducible decomposition.

4.5 Irreducible decompositions of binomial ideals

This section demonstrates how to produce an irreducible decomposition of any given binomial ideal. We first define the irreducible closure of a binomial ideal (Definition 4.5.1). Unlike a binoccular closure (Definition 4.3.4), which may have non-binomial elements in its socle, the cogenerator of a coprincipal binomial ideal is the only socle element that survives to the irreducible closure.

Definition 4.5.1. Fix a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ cogenerated by $w \in Q$. Let $R_P = \Bbbk[Q_P]/I_P$, and let $G_P \subset Q_P$ denote the group of units. Let $\overline{w}^\perp$ denote the unique graded $\Bbbk$-vector subspace of R_P such that

$$R_P = (\Bbbk[G_P] \cdot \mathbf{x}^{\overline{w}}) \oplus \overline{w}^\perp.$$

Let $\overline{w}^\perp_\infty$ denote the largest $\Bbbk[Q_P]$-submodule of R_P that lies entirely in $\overline{w}^\perp$, and set $\overline{R}_P = R_P/\overline{w}^\perp_\infty$. The *irreducible closure* of I is the ideal $\mathrm{Irr}(I) = \ker(\Bbbk[Q] \to \overline{R}_P)$.

Example 4.5.2. Let $I = \langle x^2y - xy^2, x^3, y^3, z^3\rangle$ and $\mathfrak{m}_P = \langle x, y, z\rangle$. Then $z^2(x^2 + y^2 - xy) \in \mathrm{soc}_P(I)$, so $z(x^2 + y^2 - xy)$ lies in $\mathrm{soc}_P(\langle z^2(x^2 + y^2 - xy)\rangle + I)$ but outside of $\mathrm{soc}_P(I)$. Continuing yields the irreducible closure $\mathrm{Irr}(I) = \langle x^2 + y^2 - xy\rangle + I$ of I.

Recall that a submodule N of a module M is *essential* if it intersects every nonzero submodule of M nontrivially.

Lemma 4.5.3. *If $I \subset \Bbbk[Q]$ is a P-coprincipal binomial ideal with monomial cogenerator $\mathbf{x}^w$, then the $\Bbbk[Q_P]$-submodule $\langle \mathbf{x}^{\overline{w}}\rangle = \Bbbk[G_P] \cdot \mathbf{x}^{\overline{w}}$ is essential in $\overline{R}_P$.*

Proof. Equality follows because $\mathbf{x}^{\overline{w}}$ is annihilated by $\mathfrak{m}_P$. Essentiality follows from the fact that $\overline{w}^\perp_\infty$ is the largest $\Bbbk[Q_P]$-submodule of $\overline{w}^\perp$, since any $\Bbbk[Q_P]$-submodule trivially intersecting $\langle \mathbf{x}^{\overline{w}}\rangle$ must lie entirely inside of $\overline{w}^\perp$ by definition. □

Proposition 4.5.4. *Fix a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ with monomial cogenerator $\mathbf{x}^w$. The associated primes of R_P, $\overline{R}_P$, and $R_P/\mathfrak{m}_P$ coincide.*

Proof. The associated primes of R_P and $R_P/\mathfrak{m}_P$ coincide by (Kahle and Miller, 2013, Corollary 15.2), which states that the associated primes of a mesoprimary ideal are exactly the associated primes of its associated mesoprime. By Lemma 4.5.3, $\langle \mathbf{x}^{\overline{w}} \rangle$ is an essential submodule of both R_P and $\overline{R}_P$, so their associated primes coincide. $\square$

Compare the next result to Theorem 2.3.10 for coprincipal ideals.

Theorem 4.5.5. *The irreducible closure* $\mathrm{Irr}(I)$ *of any coprincipal ideal* I *has a unique minimal primary decomposition, each component of which is irreducible.*

Proof. By Proposition 4.5.4, every associated prime of $\mathrm{Irr}(I)$ is minimal, from which the first statement follows. Moreover, by (Bass, 1962, Corollary 1.3) localization preserves essentiality, so for each $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Irr}(I))$, the ordinary localization $\langle \mathbf{x}^{\overline{w}} \rangle_{\mathfrak{p}}$ at the prime $\Bbbk[Q]$-ideal $\mathfrak{p}$ is an essential submodule of $(\overline{R}_P)_{\mathfrak{p}}$. This means $\mathrm{Irr}(I)_{\mathfrak{p}}$ has simple socle, so $\mathrm{Irr}(I)$ is irreducible by Lemma 4.1.2. $\square$

We now extend Theorems 4.3.10 and 2.2.15 to irreducible closures before stating Corollary 4.5.7, our main result for this section.

Theorem 4.5.6. *Every binomial ideal* I *is the intersection of the irreducible closures of the coprincipal components in any coprincipal decomposition of* I*.*

Proof. Fix a monoid prime $P \subset Q$ and some nonzero $f \in \mathrm{soc}_P(I)$. By Lemma 4.3.9, some nonzero monomial $\lambda \mathbf{x}^w$ of f is an I_P-witness for P. Every monomial of f that is nonzero modulo $\mathrm{Irr}(W_w^P(I))_P$ lies in the submodule $\langle \mathbf{x}^{\overline{w}} \rangle$ of $\Bbbk[Q]_P/\mathrm{Irr}(W_w^P(I))_P$, so f is nonzero modulo $\mathrm{Irr}(W_w^P(I))_P$. Lemma 4.3.8 completes the proof. $\square$

Corollary 4.5.7. *Fix a binomial ideal* $I \subset \Bbbk[Q]$*. An irreducible decomposition of* I *results by writing* I *as an intersection coprincipal components, subsequently replacing each component with its irreducible closure, and finally taking the canonical primary decomposition of each resulting component.*

Proof. Apply Theorem 4.5.6, then Theorem 4.5.5. $\square$

5

Mesoprimary modules

Mesoprimary decomposition of monoid congruences is designed to parallel primary decomposition of ideals over a commutative Noetherian ring R, with associated prime congruences (Definition 2.2.9) playing the role of prime ideals. Just as primary decomposition of ideals in R generalizes to primary decomposition of finitely generated R-modules, mesoprimary decomposition of monoid congruences, along with its notion of associated prime congruences, should generalize to congruences on monoid modules (Definition 5.1.1). The following problem appeared as (Kahle and Miller, 2013, Problem 17.11), and serves to motivate the results in this chapter.

Problem B. Generalize mesoprimary decomposition of monoid congruences to congruences on monoid modules.

In this chapter, we introduce the category Q-Mod of modules over a monoid Q (Definition 5.1.1) and generalize nearly every result from Section 2.2 to this setting. We define primary and mesoprimary congruences on arbitrary monoid modules (Definition 5.2.2) in a manner that generalizes the existing notion for monoid congruences, and give equivalent conditions for these congruences (Theorems 5.2.6 and 5.2.10) in terms of associated objects, analogous to Theorems 2.2.8 and 2.2.10. We then construct a coprincipal decomposition for any monoid module congruence with one component per key witnesses (Theorem 5.3.5), analogous to Theorem 2.2.15, which only identifies the nil elements of its quotient (see Remark 5.3.7). The resulting theory completely answers Problem B.

5.1 The category of monoid modules

In this section we define the category Q-Mod of modules over a given monoid Q and extend some of the fundamental concepts and results from monoid ideals and congruences to monoid modules. First, we record some preliminary definitions (see Grillet (2007)).

Definition 5.1.1. Fix a commutative monoid Q.

1. A *Q-module* $(T, \cdot)$ is a set T together with a left action by Q that satisfies $0 \cdot t = t$ and $(q + q') \cdot t = q \cdot (q' \cdot t)$ for all $t \in T$, $q, q' \in Q$. A subset $T' \subset T$ is a *submodule* of T if it is closed under the Q-action, that is, $Q \cdot T' \subset T'$. The submodule of T *generated by elements* $t_1, \ldots, t_r \in T$ is $\langle t_1, \ldots, t_r \rangle = \bigcup_{i=1}^{r} Q \cdot t_i$.

2. A map $\psi : T \to U$ between Q-modules T and U is a *Q-module homomorphism* if $\psi(q \cdot t) = q \cdot \psi(t)$ for all $t \in T, q \in Q$. The set of Q-module homomorphisms from T to U is denoted by $\mathrm{Hom}_Q(T, U)$, and is naturally a Q-module with action $q \cdot \psi$ given by $(q \cdot \psi)(t) = \psi(q \cdot t)$.

3. The *category of Q-modules*, denoted Q-Mod, is the category whose objects are Q-modules and whose morphisms are Q-module homomorphisms.

Example 5.1.2. Let $Q = \mathbb{N}^2$, $I = \langle x^2, y^2 \rangle$, $R = \Bbbk[Q]/I$, and

$$M = (R \oplus R)/\langle xye_1 - xye_2 \rangle,$$

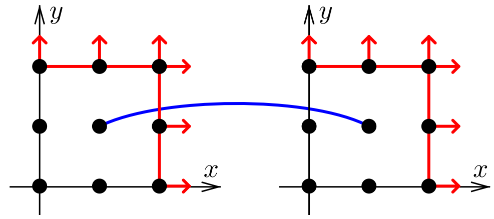


FIGURE 5.1: The monoid module that grades M in Example 5.1.2.

where e_1 and e_2 generate the free module $R \oplus R$. R is graded by the quotient monoid $Q/{\sim_I}$, and M is graded by two disjoint copies of $Q/{\sim_I}$ with both copies of xy and the nil merged. Unlike the monoid that grades R, this grading does not have a natural monoid structure. It does have a natural action by Q, correponding to the action on M by monomials in $\Bbbk[Q]$. This Q-module is depicted in Figure 5.1.

Direct sums, direct products, and tensor products exist in the category Q-Mod. We now state their constructions explicitly.

Definition 5.1.3. Fix two Q-modules T and U.

1. The *direct sum* $T \oplus U$ is the disjoint union $T \coprod U$ as sets, with the natural Q-action on each component.

2. The *direct product* $T \times U$ is the cartesian product of T and U as a set, with componentwise Q-action.

3. The *tensor product* $T \otimes_Q U$ is the collection of formal elements $t \otimes u$ for $t \in T$ and $u \in U$ modulo the equivalence relation generated by

$$t \otimes (q \cdot u) \sim (q \cdot t) \otimes u \text{ for } t \in T \text{ and } u \in U$$

   The action of Q is given by $q \cdot (t \otimes u) = (q \cdot t) \otimes u$ for $q \in Q$, $t \in T$ and $u \in U$.

Throughout the remainder of this section, let T denote an arbitrary Q-module.

Definition 5.1.4. A *congruence on* T is an equivalence relation $\sim$ on T that satisfies $t \sim t' \Rightarrow q \cdot t \sim q \cdot t'$ for all $q \in Q$ and $t, t' \in T$. The *quotient module* $T/\sim$ is the set of equivalence classes of T under $\sim$. The congruence condition on $\sim$ ensures that $T/\sim$ has a well defined action by Q.

Definition 5.1.5. A subset $T \subset Q$ is an *ideal* if it is a submodule of Q, that is, $Q + T \subset T$. An ideal $P \subset Q$ is *prime* if its complement in Q is a submonoid of Q.

Definition 5.1.6. Fix a prime ideal $P \subset Q$, and set $F = Q \backslash P$. The *localization of* T *at* P, denoted T_P, is the set $T \times F$ modulo the equivalence relation $\sim$ that sets $(t, f) \sim (t', f')$ whenever $w \cdot f' \cdot t = w \cdot f \cdot t'$ for some $w \in Q$. The localization Q_P is naturally a monoid, and T_P is naturally a Q_P-module. Write $t - f$ to denote the element $(t, f) \in T \times F$.

Remark 5.1.7. Any congruence $\sim$ on T induces a congruence on T_P.

Definition 5.1.8. *Green's preorder* on T sets $t \preceq t'$ whenever $\langle t \rangle \supset \langle t' \rangle$. *Green's relation* on T sets $t \sim t'$ whenever $\langle t \rangle = \langle t' \rangle$.

Green's preorder on monoids orders elements by divisibility, and this notion extends to Q-modules. Lemma 5.1.9 is the Q-module analogue of 2.1.9.

Lemma 5.1.9. *Green's relation $\sim$ is a congruence on T, and the quotient $T/\sim$ is partially ordered by divisibility.*

Proof. For $t, t' \in T$ and $q \in Q$, we can see $\langle t \rangle = \langle t' \rangle$ implies $\langle q \cdot t \rangle = \langle q \cdot t' \rangle$. Each element of the quotient $T/\sim$ generates a distinct submodule, so the divisibility preorder is antisymmetric, and thus a partial order. $\square$

We now generalize the notion of a nil element of a monoid.

Definition 5.1.10. An element $\infty \in T$ in a Q-module T is called a *nil* if it is absorbing, that is, $Q \cdot \infty = \{\infty\}$. The *basin* of a nil $\infty \in T$ is the set

$$B(\infty) = \{t \in T : qt = \infty \text{ for some } q \in Q\}$$

of elements of T that can be sent to ∞ under the action of Q. The *nil set* of T, denoted $N(T)$, is the collection of all nil elements in T.

Definition 5.1.11. Fix a subset $U \subset T$ of a Q-module T. A Q*-orbit* of U is a connected component of the undirected graph whose vertices are elements of U and whose edges connect two vertices $s, t \in U$ whenever $q \cdot s = t$ for some $q \in Q$. T is *connected* if it has at most one Q-orbit, and T is *properly connected* if $T \backslash N(T)$ has at most one Q-orbit.

Example 5.1.12. Let T and U be connected Q-modules with nils ∞_T and ∞_U, respectively. Assume $T\backslash\{\infty_T\}$ and $U\backslash\{\infty_U\}$ are nonempty. The module $(T \coprod U)/\langle \infty_T \sim \infty_U \rangle$ is connected and has a single nil, but it is not properly connected, since removing the nil produces two distinct orbits.

Remark 5.1.13. Unlike a monoid, a Q-module may have more than one nil element. However, by Lemma 5.1.14, each Q-orbit can have at most one nil element.

Lemma 5.1.14. *The basin of a nil $\infty \in T$ is the Q-orbit of T containing ∞.*

Proof. The basin of ∞ is clearly contained in its Q-orbit, and whenever $qt = s$ for $q \in Q$ and $s, t \in T$, we have $t \in B(\infty)$ if and only if $s \in B(\infty)$. □

There is also a notion of decomposition of Q-modules into indecomposables.

Lemma 5.1.15. *Every Q-module T has a unique decomposition $T = \bigoplus_i T_i$ as a direct sum of connected modules.*

Proof. Any Q-module is the disjoint union of its Q-orbits. □

Remark 5.1.16. Kernels, in the categorical sense, do not exist in the category Q-Mod. However, there is still a notion of kernel of a Q-module homomorphism as a congruence; see Definition 5.1.17. This definition is justified by Lemma 5.1.18, an analogue for Q-modules of the first isomorphism theorem for groups.

Definition 5.1.17. Fix a homomorphism $\phi : T \to U$. The *kernel of* ϕ, denoted $\ker(\phi)$, is the congruence $\sim$ on T that sets $t \sim t'$ whenever $\phi(t) = \phi(t')$ for $t, t' \in T$.

Lemma 5.1.18. *For any Q-module homomorphism $\phi : T \to U$, $T/\ker(\phi) \cong \mathrm{Im}(\phi)$.*

Proof. The homomorphism ϕ is surjective onto its image, and the quotient of T by $\ker(\phi)$ identifies elements with the same image under ϕ, which ensures the map $T/\ker(\phi) \longrightarrow \mathrm{Im}(\phi)$ is injective. $\square$

Corollary 5.1.19. *Any finitely generated Q-module T is isomorphic to a quotient of a direct sum of finitely many copies of Q.*

Proof. Fix a finitely generated Q-module $T = \langle t_1, \ldots, t_r \rangle$. Let $\phi : \bigoplus_{i=1}^{r} Q \longrightarrow T$, where the map on the ith summand is given by $Q \to \langle t_i \rangle$. This map is surjective, so by Lemma 5.1.18, $T \cong (\bigoplus_{i=1}^{r} Q)/\ker(\phi)$. $\square$

5.2 Primary and mesoprimary monoid modules

In this section, we generalize the notion of primary and mesoprimary monoid congruences to congruences on monoid modules.

Definition 5.2.1. Fix a Q-module T. For each $q \in Q$, let ϕ_q denote the map $T \xrightarrow{\cdot q} T$ given by action by q.

- An element $q \in Q$ acts *cancellatively on* T if ϕ_q is injective.
- An element $q \in Q$ acts *nilpotently on* T if for each $t \in T$, $(nq) \cdot t \in N(T)$ for some nonnegative integer n.

- An element $t \in T$ is *partly cancellative* if whenever $a \cdot t = b \cdot t \notin N(T)$ for $a, b \in Q$ that act cancellatively on T, the morphisms ϕ_a and ϕ_b coincide.

Definition 5.2.2. A Q-module T is

- *primary* if each $q \in Q$ is either cancellative or nilpotent on T.

- *mesoprimary* if it is primary and each $t \in T$ is partly cancellative.

A congruence $\sim$ on T is *primary* (respectively, *mesoprimary*) if $T/\sim$ is a primary (respectively, mesoprimary) Q-module.

Lemma 5.2.3. *Fix a congruence $\sim$ on Q. The Q-module $T = Q/\sim$ is (meso)primary in the sense of Definition 5.2.2 if and only if $\sim$ is a (meso)primary monoid congruence in the sense of Definition 2.2.1.*

Proof. For $q \in Q$ let $\overline{q}$ denote the image of q modulo $\sim$. An element $q \in Q$ acts cancellatively on T if and only if its image modulo $\sim$ is cancellative, and q acts nilpotently on T if and only if it has nilpotent image modulo $\sim$. This proves that T is a primary Q-module if and only if $\sim$ is primary as a monoid congruence. Lastly, assuming $\sim$ is P-primary, notice that for $a, b \notin P$, $\phi_a = \phi_b$ if and only if $\overline{a} = \overline{b} \in T$, so each $\overline{q} \in T$ is partly cancellative as a monoid element if and only if it is partly cancellative as an element of a Q-module. This completes the proof. □

Remark 5.2.4. One of the largest tasks in generalizing the results of Section 2.2 to Q-modules is to separate which constructions should happen in the monoid and which should happen in the module, since these coincide for congruences on Q. For instance, "cancellative" and "nilpotent" as defined in Definition 5.2.1 are properties of elements of Q, whereas "partly cancellative" is a property of elements of T. Roughly speaking, cancellative and nilpotent describe how a particular $q \in Q$ acts on different module elements, whereas partly cancellative describes how different monoid elements act on a particular $t \in T$.

Definition 5.2.5. Let T be a Q-module, $P \subset Q$ a prime ideal, and $\sim$ a congruence on T. For $t \in T$, let $\bar{t}$ denote the image of t in $\overline{T}_P$, and for $p \in P$, let $\phi_p : \overline{T}_P \to \overline{T}_P$ denote the morphism given by the action of p.

1. An element $w \in T$ is *exclusively maximal* in a set $A \subset \overline{T}_P$ if $\overline{w}$ is the unique maximal element of A under Green's preorder.

2. An element $w \in T$ with non-nil image in $\overline{T}_P$ is a $\sim$*-witness* for P if for each generator $p \in P$, the class of $\overline{w}$ is non-singleton under $\ker(\phi_p)$ and $\overline{w}$ is not exclusively maximal in that class.

3. An element $w' \in T$ is an *aide* for a $\sim$-witness w for P and a generator $p \in P$ if w and w' have distinct images in $\overline{T}_P$ but are not distinct under $\ker(\phi_p)$.

4. An element w with non-nil image in $\overline{T}_P$ is a *key* $\sim$*-witness for* P if $\overline{w}$ is non-singleton under $\bigcap_{p \in P} \ker(\phi_p)$ and $\overline{w}$ is not exclusively maximal in this non-singleton class.

5. The prime P is *associated to* T if T has a witness for P, or if $P = \varnothing$ and T has a Q-orbit with no nil.

Theorem 5.2.6. *A finitely generated Q-module T is primary if and only if it has exactly one associated prime ideal.*

Proof. Suppose T is primary. The set of elements with nilpotent action on T is a prime ideal $P \subset Q$. Since P is finitely generated, some non-nil element $w \in T$ satisfies $P \cdot w \subset N(T)$. This means w is a witness for P, so P is associated to T. Since $Q \backslash P$ acts cancellatively on T, any prime associated to T is contained in P. Moreover, localizing T at any prime P' contained in P identifies any element $w \in T$ with the nil in its orbit, since some $p \in P \backslash P'$ gives $p \cdot w \in N(T)$. Thus, any associated prime must also contain P, which implies P is the only associated prime.

Now suppose T has only one associated prime $P \subset Q$. If $P = \varnothing$, then every element of Q acts cancellatively on T. Now suppose P is nonempty, and fix $t \in T$. The submodule $\langle t \rangle$ is isomorphic to Q modulo some congruence. Since each witness in $\langle t \rangle$ is a witness for P, $\langle t \rangle$ is P-primary by Theorem 2.2.8. This means each $p \in P$ acts nilpotently on $\langle t \rangle$ and each $f \in Q \backslash P$ acts cancellatively on $\langle t \rangle$. Since t is arbitrary, each $p \in P$ acts nilpotently on T and each $f \in Q \backslash P$ acts cancellatively on T, meaning T is P-primary. □

Lemma 5.2.7 below generalizes Lemma 2.2.4 and is central to several proofs, including Theorem 5.2.10 and Theorem 5.3.5.

Lemma 5.2.7. *Fix a connected, P-primary Q-module T, and set $F = Q \backslash P$. Let T/F denote the quotient of T by the congruence*

$$t \sim t' \text{ whenever } f \cdot t = g \cdot t' \text{ for } f, g \in F$$

Then Green's preorder on T/F is a partial order, and T/F is finite.

Proof. Since T is P-primary, the morphisms $T \xrightarrow{\cdot f} T$ are injective for all $f \in F$, so $\sim$ is a well-defined congruence. If $\langle t \rangle = \langle t' \rangle$, then $f \cdot t = t'$ and $g \cdot t' = t$ for some $f, g \in Q$. This means $f \cdot g \cdot t = t$, so f and g are not nilpotent and lie in F, meaning t and t' are identified in T/F. This proves Green's preorder is antisymmetric.

Now, the remaining statement is trivial if $P = \varnothing$, so suppose P is nonempty. T must have a nil ∞ since Q contains elements with nilpotent action on T. The image of ∞ in T/F remains nil as well. Thus, since Q and T are both finitely generated, T/F must be finite. □

Definition 5.2.8. Fix a Q-module T, a monoid prime $P \subset Q$, and a non-nil $w \in T$.

1. Let $G_P \subset Q_P$ denote the unit group of Q_P, and let $K_q^P \subset G_P$ denote the stabilizer of $\overline{w} \in T_P$ under the action of G_P.

2. Let $\approx$ denote the congruence on Q_P that sets $a \approx b$ whenever

   (a) a and b lie in P_P, or

   (b) a and b lie in G_P and $a - b \in K_q^P$.

3. The P*-prime congruence* of T at w is given by $\ker(Q \to Q_P/\approx)$.

4. The P-prime congruence at w is *associated to* T if w is a key witness for T.

Remark 5.2.9. Definition 5.2.8 is forced to make another distinction between T and Q: should an associated prime congruence of T be a congruence on T or on Q? Theorem 2.2.10 describes P-mesoprimary congruences $\sim$ in terms of the congrunece on $Q \backslash P$ induced by its action on $Q/\sim$. The partly cancellative condition (a condition on elements of T) ensures that each $t \in T$ induces the same congruence.

Next, we give some alternate characterizations of mesoprimary Q-modules.

Theorem 5.2.10. *For a Q-module T, the following are equivalent.*

(1) T is mesoprimary.

(2) T has exactly one associated prime congruence.

(3) T is P-primary, and for $F = Q \backslash P$,

$$\ker(F \to \langle t \rangle) = \ker(F \to \langle t' \rangle)$$

for each non-nil $t, t' \in T$.

Proof. From any of these conditions, we conclude that T is primary, say with associated prime P. Notice that $\ker(F \to \langle t \rangle)$ is the prime congruence at t restricted to F. If these congruences coincide for all $t \in T$, then in particular they coincide for all witnesses, so T has exactly one associated prime congruence. This proves (3) $\Rightarrow$ (2).

Now suppose T is mesoprimary, and fix $t, t' \notin N(T)$. Then since t and t' are both partly cancellative, $a \cdot t = b \cdot t$ if and only if $a \cdot t' = b \cdot t'$ for $a, b \notin P$. This means the kernels $\ker(F \to \langle t \rangle)$ and $\ker(F \to \langle t' \rangle)$ coincide. This proves (1) $\Rightarrow$ (3).

Lastly, suppose T has exactly one associated prime congruence, and fix $t \in N(T)$. Fix $a, b \notin P$ and let $\phi_a, \phi_b : T \to T$ denote the actions of a and b on T, respectively. By Lemma 5.1.18, $\langle t \rangle \cong Q/\sim$ for some congruence $\sim$. Since T has only one associated prime congruence, so does $\sim$, so by Theorem 2.2.10 $\sim$ is mesoprimary. This means $a \cdot t = b \cdot t$ if and only $a \cdot w = b \cdot w$ for any witness $w \in \langle t \rangle$. Since T has only one associated prime congruence, these actions also coincide for all witnesses in T, meaning $\phi_a = \phi_b$. This proves (2) $\Rightarrow$ (1), thus completing the proof. $\square$

5.3 Mesoprimary decomposition of monoid modules

Definition 5.3.1. Fix a Q-module T. A *cogenerator* of T is a non-nil element $t \in T$ with $q \cdot t \in N(T)$ for every nonunit $q \in Q$. A Q-module T is *coprincipal* if it is P-mesoprimary and all its cogenerators lie in the same Green's class in T_P. A congruence $\sim$ on T is *coprincipal* if $T/\sim$ is a coprincipal Q-module.

Definition 5.3.2. Fix a Q-module T, a prime $P \subset Q$, and a P-witness $w \in T$. Let $\overline{q}$ denote the image of $q \in Q$ in Q_P, and $\bar{t}$ denote the image of $t \in T$ in T_P.

- The *order ideal $T_{\preceq w}^P$ cogenerated by w at P* consists of those $a \in T$ whose image $\overline{a} \in T_P$ precedes $\overline{w}$ under Green's preorder.

- The *congruence cogenerated by w along P* is the equivalence relation $\sim_w^P$ on T that sets all elements outside of $T_{\preceq w}^P$ equivalent and sets $a \sim_w^P b$ whenever $\overline{a}$ and $\overline{b}$ differ by a unit in T_P and $q \cdot \overline{a} = q \cdot \overline{b} = \overline{w} \in T_P$ for some $q \in Q_P$.

Lemma 5.3.3 justifies the nomenclature in Definition 5.3.2.

Lemma 5.3.3. *The congruence cogenerated by w along P is a coprincipal congruence on T cogenerated by w. Furthermore, $T/{\sim_w^P}$ is properly connected, and if $T \backslash T_{\preceq w}^P$ is nonempty, then it is the nil class of $T/{\sim_w^P}$.*

Proof. Let $T' = T/{\sim_w^P}$. Every non-nil element of T' has the image of w as a multiple, so T' is properly connected, and it is clear that the image of $T \backslash T_{\preceq w}^P$ is nil modulo $\sim_w^P$ as long as it is nonempty. Furthermore, w cogenerates $\sim_w^P$ since the result of acting by any $p \in P$ lies outside $T_{\preceq w}^P$, and any $t \in T$ with non-nil image in T' satisfies $q \cdot t = w$ for some $q \in Q$, so every cogenerator for $\sim_w^P$ lies in the Green's class of w in T_P.

It remains to show that T' is mesoprimary. By Lemma 5.2.7, $T_{\preceq w}^P$ has finitely many Green's classes in T_P, so each $p \in P$ acts nilpotently on T' and thus T' is P-primary. Furthermore, for each $t \in T$ and for $a, b \in Q \backslash P$, we have $a \cdot t \sim_w^P b \cdot t$ if and only if $a \cdot w \sim_w^P b \cdot w$. In particular, the P-prime congruences at the non-nil elements of T' coincide, so by Theorem 5.2.10, T' is mesoprimary. $\square$

Definition 5.3.4. Fix a Q-module T and a congruence $\sim$ on T.

1. An expression $\sim \, = \bigcap_i \sim_i$ of $\sim$ as the common refinement of finitely many mesoprimary congruences is a *mesoprimary decomposition* if, for each component $\sim_i$ with associated prime ideal $P \subset Q$, the P-prime congruences of $\sim$ and $\sim_i$ at each cogenerator for $\sim_i$ coincide.

2. A mesoprimary decomposition $\sim \, = \bigcap_i \sim_i$ is *key* if, for each P-mesoprimary component $\sim_i$, every cogenerator for $\sim_i$ is a key P-witness for $\sim$.

We are now ready to give the main result of this section. Theorem 5.3.5 implies, as a special case, that every monoid module with at most one nil element admits a key mesoprimary decomposition (see Remark 5.3.7).

Theorem 5.3.5. *Fix a congruence $\sim$ on a Q-module T. The common refinement of the coprincipal congruences cogenerated by the key witnesses of $\sim$ identifies only the nil elements of $T/\sim$.*

Proof. The nil class of the congruence cogenerated by a witness $w \in T$ for P contains the nil in the connected component of w (if one exists), as well as every element outside of this connected component. This means any P-coprincipal component identifies all of the nil elements of T.

Now, fix distinct $a, b \in T$ and assume a is not nil. If a and b lie in distinct connected components, then any cogenerated congruence whose order ideal contains a does not identify a and b. Assuming a and b lie in the same connected component, it suffices to find a monoid prime $P \subset Q$ and a key witness $w \in T$ for P such that a and b are not equivalent under $\sim_w^P$. Fix a prime P minimal among those containing the ideal $I = \{q \in Q : q \cdot a = q \cdot b\}$. Notice that I (and thus P) must be nonempty since a and b lie in the same connected component.

Since P contains I, the elements a and b have distinct images $\overline{a}$ and $\overline{b}$ in T_P, and each $\overline{q} \in I_P$ also satisfies $\overline{q} \cdot \overline{a} = \overline{q} \cdot \overline{b}$. By minimality of P_P over I_P, there is a maximal Green's class among the elements $\{\overline{q} \in Q_P : \overline{q} \cdot \overline{a} \neq \overline{q} \cdot \overline{b}\}$. Pick an element $q \in Q$ such that $\overline{q}$ lies in this Green's class, and set $w = q \cdot a \in T$. Then w is a key witness for P by construction, and the localization of $\sim_w^P$ does not equate $\overline{a}$ and $\overline{b}$ in T_P, so $\sim_w^P$ does not equate a and b in T. This completes the proof. $\square$

Corollary 5.3.6. *Fix a Q-module T and a congruence $\sim$ on T. If $T/\sim$ has at most one nil element, then $\sim$ admits a key mesoprimary decomposition.*

Proof. Apply Theorem 5.3.5 and Lemma 5.3.3 to $T/\sim$. $\square$

Remark 5.3.7. Theorem 5.3.5 states that mesoprimary decomposition fails to distinguish nil elements from one another, and that this is the only obstruction to constructing mesoprimary decompositions in this setting. Fortunately, for the purposes of decomposing binomial submodules of graded modules over a monoid algebra, these elements will likely already be indistinguishable, as they should all correspond to zero in the module (see Problem 6.7).

Remark 5.3.8. Many of the results from Section 3.3 regarding minimal and irredundant mesoprimary decompositions of monoid congruences likely extend to the setting of monoid module congruences as well. See Question 6.5 and Problem 6.6 for more detail.

6

Future work

Outlined below are some questions for future study. Of these, Question 6.4 and Problem 6.7 are likely the most important.

As Examples 3.2.3 and 3.3.5 demonstrate, identifying classes of redundant components in mesoprimary decompositions becomes more subtle if components that are not induced by witnesses are allowed.

Problem 6.1. Is there a class of congruences that can be omitted from every (not necessarily induced) mesoprimary decomposition for a given monoid congruence?

Problem 6.2. Is there a class of ideals that can be omitted from every (not necessarily induced) mesoprimary decomposition for a given binomial ideal?

Examples 4.4.1 and 4.4.3 exhibit binomial ideals that do not admit binomial irreducible decompositions. Problem 6.3 and Question 6.4 ask for which binomial ideals this occurs. Question 6.4 is more general than Problem 6.3, but may involve primary decompositions that do not arise from mesoprimary decomposition.

Problem 6.3. Determine when all of the binoccular components in the decomposition from Theorem 4.3.10 without simple socle can be omitted.

Question 6.4. Which binomial ideals admit binomial irreducible decompositions?

Now that mesoprimary decomposition of monoid congruences has been extended to congruences on monoid modules (Chapter 5), it is natural to ask if the results from Chapter 3 also extend to this setting.

Question 6.5. Which associated prime congruences of a given Q-module T appear as the associated prime congruence of some mesoprimary component in every mesoprimary decomposition of T?

Problem 6.6. Determine when a congruence $\sim$ on a given Q-module T admits a unique minimal induced mesoprimary decomposition, and when $\sim$ admits a unique irredundant induced coprincipal decomposition.

Mesoprimary decomposition of congruences on a monoid Q is the combinatorial side of decomposing binomial ideals in the corresponding monoid algebra $\Bbbk[Q]$. Likewise, mesoprimary decomposition in the categorical framework presented in Chapter 5 for Q-modules should lift to a category of $\Bbbk[Q]$-modules that are finely graded by Q-modules. Problem 6.7 was originally given as (Kahle and Miller, 2013, Problem 17.13), but relied on an answer to Problem B. Now that a solution to Problem B has been presented, Problem 6.7 can be stated more concretely.

Problem 6.7. Extend mesoprimary decomposition of binomial ideals in $\Bbbk[Q]$ to "binomial submodules" in a category of $\Bbbk[Q]$-modules graded by Q-modules.

Bibliography

Bass, H. (1962), "Injective dimension in Noetherian rings," *Trans. Amer. Math. Soc.*, 102, 18–29.

Bayer, D. and Sturmfels, B. (1998), "Cellular resolutions of monomial modules," *J. Reine Angew. Math.*, 502, 123–140.

Bayer, D., Peeva, I., and Sturmfels, B. (1998), "Monomial resolutions," *Math. Res. Lett.*, 5, 31–46.

Eisenbud, D. (1995), *Commutative algebra*, vol. 150 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, With a view toward algebraic geometry.

Eisenbud, D. and Sturmfels, B. (1996), "Binomial ideals," *Duke Math. J.*, 84, 1–45.

Grillet, P. A. (2001), *Commutative semigroups*, vol. 2 of *Advances in Mathematics (Dordrecht)*, Kluwer Academic Publishers, Dordrecht.

Grillet, P. A. (2007), "Commutative actions," *Acta Sci. Math. (Szeged)*, 73, 91–112.

Haarmann, J., Kalauli, A., Moran, A., O'Neill, C., and Pelayo, R. (2013), "Factorization properties of Leamer monoids," *Semigroup Forum (accepted)*, arXiv:1309.7477 [math.AC].

Hochster, M. (1977), "Cohen-Macaulay rings, combinatorics, and simplicial complexes," in *Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975)*, pp. 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26, Dekker, New York.

Kahle, T. and Miller, E. (2013), "Decompositions of commutative monoid congruences and binomial ideals," *Preprint*, arXiv:1107.4699 [math.AC].

Miller, E. and Sturmfels, B. (2005), *Combinatorial commutative algebra*, vol. 227 of *Graduate Texts in Mathematics*, Springer-Verlag, New York.

Noether, E. (1921), "Idealtheorie in Ringbereichen," *Math. Ann.*, 83, 24–66.

O'Neill, C. and Pelayo, R. (2013), "On the Linearity of ω-primality in Numerical Monoids," *Journal of Pure and Applied Algebra (accepted)*, arXiv:1309.7476 [math.AC].

Vasconcelos, W. V. (1998), *Computational methods in commutative algebra and algebraic geometry*, vol. 2 of *Algorithms and Computation in Mathematics*, Springer-Verlag, Berlin, With chapters by David Eisenbud, Daniel R. Grayson, Jürgen Herzog and Michael Stillman.

Biography

Christopher David O'Neill was born on October 16th, 1985 in San Mateo, California. He earned a B.A. in Mathematics and a B.S. in Computer Science from San Francisco State University in the Spring of 2009, and a Ph.D in Mathematics from Duke University in the Spring of 2014.

While at Duke University, Christopher received the L.P. and Barbara Smith Award for Teaching Excellence in September of 2012. He also worked as a Research Assistant for the Pacific Undergraduate Research Experience in the Summer of 2012, and again in the Summer of 2013.

At the time of his defense, Christopher has two publications:

- *Factorization properties of Leamer monoids* (Haarmann et al. (2013)). This is an undergraduate research project from PURE Math 2013, coadvised with Roberto Pelayo.

- *On the linearity of ω-primality in numerical monoids* (O'Neill and Pelayo (2013)).

Upon graduating from Duke University, Christopher will begin a three-year Post-Doc position at Texas A&M University in the Fall of 2014 after working as a Research Assistant for the San Diego State Mathematics Research Experience in Mathematics in the Summer of 2014.